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Shock waves in solids

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SHOCK WAVES IN SOLIDS

by

William Walter Predebon

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
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LIST OF SYMBOLS

$x_i = (x_1, x_2, x_3)$	Fixed rectangular cartesian spatial coordinate system
$X_A = (X_1, X_2, X_3)$	Material coordinates
$u^\alpha = (u^1, u^2)$	General curvilinear system of surface coordinates
$i, j, k, \dots = 1, 2, 3$	Cartesian tensor indices - spatially dependent variables
$A, B, C, \dots = 1, 2, 3$	Cartesian tensor indices - materially dependent variables
$\alpha, \beta, \gamma, \dots = 1, 2$	General curvilinear tensor indices
$\Sigma(t)$	Moving singular surface (t is time)
$[Z]$	Jump in a field variable z across the shock-front
$\frac{\delta}{\delta t}$	Delta time derivative
$\frac{d}{dt}$	Material derivative or total derivative
G	Spatial normal speed of propagation of the shock wave
D	Material normal speed of propagation of the shock wave
$u_i = (u_1, u_2, u_3)$	Displacement vector
$v_i = (v_1, v_2, v_3)$	Velocity vector
$n_i = (n_1, n_2, n_3)$	unit normal vector
$\xi_1 = \xi = (\xi_1, \xi_2, \xi_3)$ $= (\xi, \eta, \zeta)$	Discontinuity vector in the gradient of the displacement vector
$g_{\alpha\beta}$	$= x_{i,\alpha} x_{i,\beta}$: First fundamental form of the wave-front

$b_{\alpha\beta}$	$=x_{i,\alpha\beta}n_i$: Second fundamental form of the wave-front
$e_{ij}=\tilde{e}$	Spatial strain tensor (Eulerian)
I,II,III,IV,V	Strain invariants (appropriate to the problem)
$e_a=\tilde{e}$	Principal strains ($a=1,2,3$)
$h_a=\tilde{h}$	Hencky strain measure ($a=1,2,3$)
$t_{ij}=\tilde{t}$	Spatial stress tensor
$t_a=\tilde{t}$	Principal stresses ($a=1,2,3$)
ℓ_i	Principal directions of the stress and strain tensors
λ_a	Principal stretches ($a=1,2,3$)
ρ	Current density in the deformed medium
ρ_0	Density of the initially strained medium
ρ_{00}	Density of the unstrained natural state
τ	Reciprocal of the current density
λ, μ	Second-order elastic constants (elasticities)
ℓ, m, n	Third-order elastic constants (elasticities)
p	Fourth-order elasticity
e	Internal energy function per unit mass
U	Internal energy function per unit undeformed volume
S	Entropy per unit mass
θ	Temperature absolute
P	Pressure

$\det(e_{ij}) = e_{ij} $	Determinant of e_{ij}
J	Jacobian of transformation
$f_\alpha, \alpha=1,2,\dots,n$	Thermodynamic tensions
$v_\alpha, \alpha=1,2,\dots,n$	Thermodynamic substate variables

Note: Same letters used in different contexts give corresponding meanings; e.g., u^α , u_i , etc. Symbols used once only or where meaning is clear from context are not included here.

1. INTRODUCTION

Compared to the long established status of the shock wave model in gas-dynamics [1]¹, studies of shock phenomena in solids are quite recent and few. Though nonlinearity of elastic response in static problems has received much acceptance in the last fifty years, studies of nonlinear wave motion has been quite recent and scarce. Adequacy of linear theory to explain a large body of experimental data may be one reason and plastic response of solids to high stresses may be another reason. However, recent demands of technology [2] and improved techniques of observation and experimentation [3, 4, 5] have attracted a great deal of attention to the study of shock waves in solids. It is recognized that in solids a strong impact or explosion produces an elastic shock followed by a plastic one [4-7]. Thus a complete understanding of flow features in solids requires a much wider study of purely elastic, elastic-plastic and plastic shock waves. This study will concern itself with only the first case; i.e., purely elastic shock waves in solids.

However, for the elastic shock in a solid, it appears that the gas-dynamical model is adopted as it stands [2-5, 8-12]. An exact study of this model should be based on

¹Numbers in square brackets refer to literature cited in the bibliography.

the study of shock waves in a hyperelastic medium rather than based on a gas-dynamical model. Indeed, the present study not only reveals a number of apparent analogies between the pure longitudinal shock in a hyperelastic medium and the gas-dynamical one, it also points out the differences that need clarification.

A number of recent studies in the field of shock waves in solids use the material description [13-23]. It is first stressed here that such a description needs a number of clarifications; and even after establishing a consistency, some of the final results can be physically meaningful only if transformed to a spatial description. After explaining this view, a complete spatial formulation is given for a hyperelastic medium, which itself has a few novel features. The formulation is given for both an isotropic and anisotropic hyperelastic medium.

Using this formulation, the possible types of shock waves are discussed. Since the shock conditions do not provide a unique solution, one imposes admissibility conditions. Several admissibility conditions are found in the literature. These are briefly discussed here. The one imposed in this study is the requirement that entropy must increase across the shock wave. However, more detailed study is difficult at the present time due to the lack of knowledge about the form of the internal energy function,

which is needed for the study of genuinely strong shock waves. So the study is limited at the present time to shock waves of small but finite amplitude (strength), for which entropy changes are obtained.

2. CONCEPT OF A SHOCK WAVE AND THE THEORY OF SINGULAR SURFACES

2.1. General Considerations

Shock wave phenomena in gases have a century old history. The advent of fast moving airplanes on the one hand and astrophysical observations and explorations on the other hand, have increased the importance of a shock wave model to explain a large number of phenomena. It is important to realize that any theory produces only models based on idealizations to explain a complicated physical phenomenon. Therefore models are based not only on ideal materials but on idealizations of the situation itself. The success of a model, as measured by acceptability, depends on the reasonableness of the explanations it produces for observations on the one hand and the predictions it can make on the other. Thus the success of a model may be temporary; novel observations may in turn prove it to be inadequate; further, increased refinement of observations may also prove it to be insufficient. Consequently seeking a proof for the proposition that a model does describe a particular phenomenon is an irrelevant inquiry.

From this viewpoint the shock wave model has gained complete respectability in gas-dynamics. Once a model is adopted, its inevitable mathematical consequences produce predictions which are crucial in determining the reasonable-

ness of the model.

In contrast the idea of a shock wave model in solids is not quite as common. Studies of nonlinear wave motion has been scarce. Adequacy of linear theory to explain a large number of observations of phenomena of interest may be one reason or perhaps the observational limitations have not demanded models besides those based on the linear theory. In addition, the differences in the response of solids and gases may also be a reason for this lack of interest.

However, the last decade has seen a marked change in viewpoint as evidenced by a large number of papers, both theoretical and experimental in character, on shock waves in solids. In spite of this increased interest, the models in solids still appear to be quite ad hoc in nature. These are, of course, based on physical considerations which do remain crucial and decisive in all physical theories. The attempt of this present study will be to seek a basis for such ad hoc models starting with some general theories suitable for any continuum.

The behavior of solids is taken to be governed by elastic response at lower stress levels and by plastic response at higher stress levels. One group of studies rely on this model [4, 5, 6, 7]. Then there is another group of studies that adopt the gas-dynamical model for a solid [2-5, 8-12], which is more akin to the nonlinear elastic

model that forms the basis of the present study. The following appears to be an acceptable description for the generation of a shock wave: the result of an explosion or a strong impact at the end face of a semi-infinite medium is to produce first an elastic shock followed then by a plastic shock. The latter model (gas-dynamic or nonlinear elastic) can be assumed to describe the elastic shock. The plastic shock should be characterized by different constitutive laws behind and ahead of the shock wave, with possibly another shock wave with a plastic state on both sides. Thus far studies of shock waves with different constitutive laws for the medium ahead and behind the shock wave appear to be rare in the literature (except in magneto-gas-dynamics). However the study of the first shock wave (elastic) based on the "fluid model" appears to have received much attention. It is believed that the major reason for the acceptance of this model is due to the large amount of theoretical studies provided from the literature on gas-dynamics.

But such a fluid model should follow from the exact nonlinear elastic theory. One finds no such question posed in the literature. The only explanation given seems to be that for strong shocks, shear is unimportant and hence the gas-dynamical model is reasonable. It is not at all clear how to arrive at this model from the nonlinear elastic model.

The attempt of the present study will be first to

formulate an exact theory of an elastic shock based on the equations of hyperelasticity. It is then indicated that, though differing radically from the gas-dynamical model, in certain cases one can produce a model formally comparable to the gas-dynamical model adopted.

In conclusion, when the term shock wave is used in this study what is meant is an elastic shock wave. The more difficult study of a shock wave separating an elastic-plastic region is not pursued here.

2.2. Concept of a Shock Wave

It may first be necessary to note that the concept of a wave as being associated with a wave number and a frequency is valid only for periodic phenomena in linear, nondissipative and nondispersive systems. For pulse propagation problems, which is more akin to the shock wave on the one hand, it is invoked that a wave consists of all possible wavelengths and frequencies, an appropriate superposition of which can yield the pulse in linear theory. The same ideas enable one to extend this concept of a wave to dissipative and dispersive cases; all this is carried out via Fourier and/or Laplace transform techniques. For a nonlinear wave these ideas can only have a qualified meaning, if one at all. Further transform techniques used in the linear theory are not useful in the nonlinear theory. The basic feature

of a simple nonlinear system is the generation of different wave numbers and frequencies when excited by a given wave number and frequency. Studies in nonlinear wave theory do exist in the literature in the field of hydrodynamic stability. However, the concepts used there are difficult to follow and interpret for the current study. The only way of interpreting a wave that is perhaps close to the present view is that of characteristic theory. The concept of a singular surface theory, which is presented here (which includes characteristics in all known cases), seems to be a logical starting point.

Usage gives meaning to a term; as such the shock wave has come to mean in fluid mechanics a discontinuity surface across which field variables themselves are discontinuous; where as the sonic wave is one across which their derivatives are discontinuous; here density, velocity vector and internal energy provide the basic field variables. Further, the gas-dynamical system is a first order system (neglect of viscosity, heat conduction, and other dissipative and dispersive quantities is assumed throughout except where particularly mentioned). Thus a sonic wave is given by characteristic theory. In elasticity the basic variables are the displacement components (or an equivalent) in terms of which density, velocity and strain are given. If the displacement vector is taken as the set of basic variables, then it can-

not be discontinuous since this indicates separation of the material. It is only their derivatives that can be discontinuous; which in turn causes density, velocity, strain, stress, etc. to be discontinuous since they are functions of the displacement gradients. An important property of a shock wave is the dependence of its normal speed of propagation on its amplitude; of course this can be true only for a nonlinear system. For a linear system, the speed is constant. This amplitude dependence for a nonlinear system may characterize all shocks,² including the ones when the basic system is not necessarily a first order system.

It is desired to identify a singular surface with a wave. A wave is basic to all communications; the transfer of all messages is through wave propagation. A message is a disturbance of a field propagating through space. For all field theories, the speed of propagation of a disturbance is finite. Thus a disturbance arising at a point is not felt instantaneously at all points in the medium. Further such a concept of a wave is only meaningful for a hyperbolic system. Since waves are carriers of messages, for a given medium, there must be a region where a message has not arrived and another where it has; the boundary separating these disturbed and undisturbed regions is itself a wave. This is the singular surface.

²Shock wave, shock and shock-front are used interchangeably to mean the same.

Further since singular surface theory is used, the study is therefore limited to the head of the disturbance. Even so since the head of the disturbance is the strongest part of the wave, the study provides much information. In addition the governing system of equations should not have higher derivatives representing dissipation and dispersion so as to disallow the existence of real characteristics. Thus the method is always applicable for a nondissipative and nondispersive medium and for a general medium this may give a first approximation.

Mathematically the shock wave has been idealized as a discontinuity surface, i.e., a surface across which basic field variables and/or their derivatives suffer finite jumps. Physically all real materials do not admit such sharp discontinuities, such as a viscous gas. Shock waves exist only in ideal materials. For real materials, some dissipation is always present. So that field variables, instead of suffering discontinuous changes, suffer rapid changes in small distances. This leads to the notion of shock wave thickness which comes under the study of the structure of the shock wave. The shock wave thickness is controlled by dissipative structural mechanisms, such as viscosity, heat conduction, etc. However, this problem is not pursued in this study. Recently, however, Bland [24] investigated the shock structure for a plane longitudinal shock wave.

2.3. Singular Surface Theory and Compatibility Conditions: Spatial Formulation

As proposed earlier, the appropriate framework within which a logically correct study of waves, as understood here, is to be made, is that of a singular surface. Thus the two terms, wave and singular surface, are used to mean one and the same throughout the present study.

A singular surface is a surface across which at least some of the field variables or their derivatives are discontinuous. If the location (and possibly configuration) in space changes with time then it is called a propagating singular surface. The speed of such a propagating surface normal to itself will be called the normal speed of propagation. The speed will always be denoted by G in the spatial formulation. The normal speed may be an absolute constant depending on the material properties of the medium ahead (of the shock wave); then the propagation is called isotropic, homogeneous [25]; if it depends on the normal vector but independent of position and place, the propagation is anisotropic, homogeneous; if it depends on positional coordinates, it is nonhomogeneous. The normal speed of propagation of all waves designated as weak waves, will be one of those types above. But shock waves, which are not weak waves, are characterized by the main property that its normal speed of propagation G depends on the amplitude of the shock wave!

Let the coordinates x_i denote the spatial location referred to a fixed rectangular cartesian system and t denote the time. Let the singular surface be represented by

$$f(x_i, t) = 0. \quad (2.3.1)$$

Cartesian tensor notation is used for the spatial system x_i ; latin indices (i, j, k, \dots) , which range over 1, 2 and 3, are used to denote vectors and tensors and partial differentiation if preceded by a comma.

Let p_i denote the gradient of f and n_i denote the unit normal to the surface so that

$$p_i = f_{,i} ; \quad p_i p_i = p^2 ; \quad p_i = p n_i, \quad (2.3.2)$$

where

$$f_{,i} = \frac{\partial f}{\partial x_i} .$$

Further a repeated index denotes summation over that index as in cartesian tensor notation. Note, the following is the convention and nomenclature used throughout this study: Lower case letters (x_i , e_{ij} , d_{ij} , etc.) will in general be used for spatial coordinates and spatially dependent variables to distinguish them from material coordinates and materially dependent variables for which upper case letters, capitals, (X_A , E_{AB} , etc.) are used. Subscripts used on spatial variables are lower case latin indices

(i,j,k,...) and subscripts used on material variables are capital letters (A,B,C,...). As a general rule, in sections where spatial and material variables appear together, the above convention will be used exclusively. However, in sections where only spatial variables appear, this rule will be relaxed since it causes no ambiguity. For example, some of these exceptions are the following spatial variables; G, U, Σ and S.

Let $P(x_i, t)$ and $Q(x_i + \Delta x_i, t + \Delta t)$ be the points on the singular surface $\Sigma(t)$ at times t and $t + \Delta t$, lying on the normal to $\Sigma(t)$; then if Δn is the distance along the normal to $\Sigma(t)$ between P and Q, one must have

$$\Delta n = G\Delta t; \quad \Delta x_i = \Delta n n_i,$$

giving

$$\frac{\delta x_i}{\delta t} = G n_i; \quad \Delta x_i = G n_i \Delta t. \quad (2.3.3)$$

This delta time derivative was introduced by Thomas [26], see also Hayes [27] and Truesdell [28, 29]; it denotes the rate of change of a quantity defined on the wave-front $\Sigma(t)$ as observed by a rider on the wave-front traveling along the normal direction; thus it is composed not only of the ordinary time rate, given by the partial derivative, but also of a convective part due to the motion of the surface. Thus the equations of $\Sigma(t)$ and $\Sigma(t + \Delta t)$ respectively are given by

$$f(x_i, t) = 0; \quad f(x_i + \Delta x_i, t + \Delta t) = 0.$$

From these one can obtain

$$\frac{\partial f}{\partial t} + G n_i \frac{\partial f}{\partial x_i} = 0. \quad (2.3.4)$$

Since however $f_{,i} = p_i = p n_i$ and by defining n_i as a unit vector, $n_i n_i = 1$, one obtains

$$G = - \frac{\partial f / \partial t}{p} = - \frac{f_t}{p}. \quad (2.3.5)$$

Thus if f is independent of t , the surface is stationary and G vanishes. For a propagating surface, $G \neq 0$ and so $f_t \neq 0$, which is assumed throughout.

For what are called weak waves in this study, the normal speed of propagation G is completely determined, in terms of n_i , x_i , and t [25, 30], by the system itself; further, from this the complete geometry together with the history of $\Sigma(t)$ is determined from what is called Ray theory [25, 30, 31].

However, for a shock wave or a strong wave, G (the normal speed of propagation of the shock) depends on the amplitude too; then Ray theory is inapplicable. Further, a shock wave has the additional property that the normal speed of propagation G is not determined by the system itself; an additional hypothesis is needed. There are only a

few physical situations where it is determined and then only from an additional physical hypothesis. The only known cases of determining G appear to be:

1. Blast wave model with an energy hypothesis [4, 32-35].
2. Blast wave model with a momentum hypothesis [2].
3. The piston problem [1, 30].
4. The Riemann problem (or the diaphragm problem) [1, 30]

Except for the last one, the other three examples of gas-dynamics do have analogies in elasticity. The last problem is one of a gas contained in box separated by a diaphragm; pressures and densities on both sides being different. As the diaphragm is broken a shock wave travels from a region of high pressure and density to that of a lower pressure and a rarefaction wave travels into the region of lower density. Simulation of this model in solids seems to be difficult experimentally and this may be the reason for it not being adopted in solid mechanics.

Now $\Sigma(t)$ divides the 3-space x_i at any time into two regions, denoted by R^+ (or R_1) and R^- (or R_0) (see Figure 2.1). Let $Z(x_i, t)$ be any field variable (scalar, vector, or tensor). $\Sigma(t)$ is defined to be singular with respect to $Z(x_i, t)$ if $Z(x_i, t)$, continuous in R^+ and R^- separately, at least in the neighborhood of $\Sigma(t)$, suffers a

discontinuity across $\Sigma(t)$; such that $Z(x_i, t)$ approaches definite limit values $Z^+(x_i, t)$ and $Z^-(x_i, t)$ as x_i approaches x_i^0 on $\Sigma(t)$ for a fixed time t , while remaining within R^+ and R^- respectively. The value of a quantity Z in the region R^- is denoted by Z^- and that in R^+ will be denoted by Z^+ . Later, for convenience, the value ahead (R^-) is denoted with a suffix zero and the value behind (R^+) is written as it stands without any suffix or \pm sign. The square bracket³ below is used to denote the jump in a quantity across $\Sigma(t)$, as

$$[Z] = Z^+ - Z^- , \quad (2.3.6a)$$

or

$$[Z] = Z - Z_0 . \quad (2.3.6b)$$

Further the following convention is adopted throughout: positive normal points into the region R^- ; G is positive for the surface moving from R^+ to R^- ; lastly $R^-(R_0)$ is the unshocked region and $R^+(R)$ is the shocked region. The functions Z^+ and Z^- are defined only in their corresponding regions, at least in a neighborhood of $\Sigma(t)$ and have continuous one-sided derivatives in the regions of their definitions.

³The square bracket is also used for citing references in the bibliography, however, context will be sufficient to prevent any ambiguities.

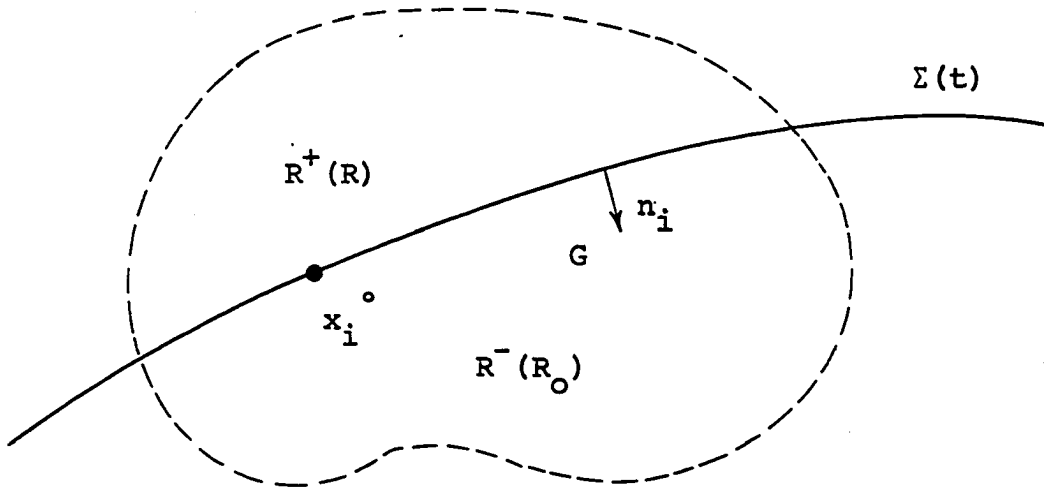


Figure 2.1. Pictorial representation of a singular surface

Consider now what are called compatibility conditions [25, 26, 28, 29, 36]. Let the discontinuities suffered by Z and its normal derivatives be denoted by

$$[Z] = A; \quad [Z_{,i}]n_i = B; \quad [Z_{,ij}]n_i n_j = C. \quad (2.3.7)$$

The object now is to express jumps in the partial derivatives of Z in terms of A , B , C and geometrical quantities associated with $\Sigma(t)$. Let $u^{\bar{\alpha}}$ ($\alpha=1,2$) be a Gaussian system of surface coordinates for $\Sigma(t)$, which are, in general, curvilinear. Let $g_{\alpha\beta}$ and $b_{\alpha\beta}$ be the first and second fundamental forms respectively, given by

$$g_{\alpha\beta} = x_{i,\alpha} x_{i,\beta}; \quad b_{\alpha\beta} = n_i x_{i,\alpha\beta}, \quad (2.3.8)$$

where $x_i = x_i(u^\alpha, t)$ is a parametric representation of $\Sigma(t)$.

Greek indices (α, β, \dots) ranging over $(1, 2)$, denote surface coordinates and a Greek letter as a suffix after a comma denotes covariant derivative, though this problem never arises in this study except for the definition of $b_{\alpha\beta}$ in Equation 2.3.8.

To obtain compatibility conditions one needs what is called Hadamard's lemma [28], stated here as: "The tangential derivative of a discontinuity across a singular surface is the same as the discontinuity in the tangential derivative". Stated mathematically it takes the form

$$\begin{aligned} [Z]_{,\alpha} &= [Z]_{,i} x_{i,\alpha} = [Z]_{,i} x_{i,\alpha} \\ &= [Z]_{,\alpha} \\ &= [Z]_{,i} x_{i,\alpha} \quad . \end{aligned} \tag{2.3.9}$$

Here $x_{i,\alpha}$ denote partial derivatives of x_i with respect to u^α and thus denote tangent vectors to the singular surface $\Sigma(t)$. In a spatial system these are always continuous and so can be taken in and out of the square bracket.

Consider now the resolution of any vector v_i into components as

$$v_i = v_n n_i + v^\alpha x_{i,\alpha} \quad . \tag{2.3.10}$$

Since $n_i x_{i,\alpha} = 0$, one has

$$v_n = v_i n_i ;$$

$$\begin{aligned} v_i x_{i,\beta} &= v^\alpha x_{i,\alpha} x_{i,\beta} \\ &= v^\alpha g_{\alpha\beta} \\ &= v_\beta \quad . \end{aligned} \tag{2.3.11}$$

Thus v_n denotes the normal component of v_i and v^α, v_α are the contravariant and the covariant tangential components respectively of v_i ; in the present study these components will be the components of v_i on the singular surface $\Sigma(t)$.

Since $[Z_{,i}]$ is a vector, one can write

$$[Z_{,i}] = a n_i + d^\alpha x_{i,\alpha} . \tag{2.3.12}$$

Multiplying Equation 2.3.12 by n_i and $x_{i,\beta}$ respectively, one obtains

$$\begin{aligned} a &= [Z_{,i}] n_i = B; \\ [Z_{,i}] x_{i,\beta} &= d^\alpha x_{i,\alpha} x_{i,\beta} \\ &= d^\alpha g_{\alpha\beta} \\ &= d_\beta \quad . \end{aligned} \tag{2.3.13}$$

Hadamard's lemma is now imposed on Equation 2.3.13 to obtain

$$\begin{aligned}
d_\beta &= [Z, {}_i x_{i,\beta}] \\
&= [Z, {}_\beta] \\
&= [Z]_{,\beta} \\
&= A_{,\beta} \quad ,
\end{aligned}$$

from which

$$d^\alpha = g^{\alpha\beta} d_\beta = g^{\alpha\beta} A_{,\beta} \quad ,$$

thus

$$[Z, {}_i] = B n_i + g^{\alpha\beta} A_{,\beta} x_{i,\alpha} \quad . \quad (2.3.14)$$

This formula is called the geometrical condition of compatibility (of the first order). Higher order conditions can be obtained by further differentiation. The remaining one, called the kinematical condition of compatibility, since it involves time, is derived more easily by use of the concept of the delta time derivative. This is, by what was explained earlier, the rate of change of a quantity defined on the wave-front as observed by a rider on the wave-front travelling along the normal direction. Thus for any quantity H defined on the wave-front,

$$\begin{aligned}
\frac{\delta H}{\delta t} &= \lim_{\Delta t \rightarrow 0} \frac{H(x_i + G n_i \Delta t, t + \Delta t) - H(x_i, t)}{\Delta t} \\
&= \frac{\partial H}{\partial t} + G n_i \frac{\partial H}{\partial x_i} \quad . \quad (2.3.15)
\end{aligned}$$

Now identify Z^+ and Z^- with H , with the interpretation that the derivatives are now one-sided derivatives, and subtract the results to obtain

$$[\frac{\partial Z}{\partial t}] = -G[Z, i]n_i + \frac{\delta [Z]}{\delta t} ,$$

or

$$[\frac{\partial Z}{\partial t}] = -GB + \frac{\delta A}{\delta t} . \quad (2.3.16)$$

Equations 2.3.14 and 2.3.16 are called first order conditions.

Relations 2.3.14 and 2.3.16 will be used in writing the shock conditions which will be derived later. Higher order conditions of compatibility are useful in obtaining vorticity changes or changes in higher order quantities across a shock. These are, for $A = [Z] = 0$, given by

$$[Z, ij] = Cn_i n_j + g^{\alpha\beta} B_{, \alpha} (n_i x_{j, \beta} + n_j x_{i, \beta}) - B b^{\alpha\beta} x_{i, \alpha} x_{j, \beta} , \quad (2.3.17)$$

$$[\frac{\partial^2 Z}{\partial x_i \partial t}] = \{\frac{\delta B}{\delta t} - GC\}n_i - (GB)_{, \alpha} g^{\alpha\beta} x_{i, \beta} , \quad (2.3.18)$$

$$[\frac{\partial^2 Z}{\partial t^2}] = G^2 C - 2G\frac{\delta B}{\delta t} - B\frac{\delta G}{\delta t} . \quad (2.3.19)$$

In obtaining the above results the formulae for Gauss, Weingarten and Thomas [26] were used. These are

respectively:

$$x_{i,\alpha\beta} = b_{\alpha\beta} n_i; \quad n_{i,\alpha} = -b_{\alpha}^{\beta} x_{i,\beta}, \quad (2.3.20)$$

$$\frac{\delta n_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta}. \quad (2.3.21)$$

Proofs of the above relations follows the same arguments used in obtaining Equations 2.3.14 and 2.3.16 and are therefore omitted. For derivations of these and higher order conditions see Nariboli [25] and Thomas [26].

2.4. Singular Surface Theory and Compatibility Conditions: Material Formulation

Studies of shock waves in fluids are mostly done in the spatial⁴ system, except in one dimensional problems and even the latter studies are quite rare. Contrarily, studies in elasticity are mostly done by use of the material system. In static problems this creates no ambiguities; further it is taken to offer an advantage. Working with the spatial system leads to an "unknown boundary-value problem". The boundary conditions are to be satisfied on the deformed configuration, which itself is unknown and is required to be found as part of the solution of the problem. These difficulties are not novel to mechanics; the problems of jets, elastic-plastic boundaries, water waves and others

⁴Spatial system, deformed system, Eulerian system and current configuration are used interchangeably to mean the same.

are examples of analagous mathematical problems. Though these difficulties have not deferred researches in other fields from using a spatial formulation, elasticity seems to be an exception. Simplicity of various formulae may be another factor contributing to the use of a material system in elasticity. Anyway it is not the object of this study to criticize the current practices. Such studies have led to a tremendous understanding of nonlinear elasticity. Ultimately the solutions are transformable into each other, if needed.

But in wave propagation problems, the use of the material system leads to quite, apparently at least, unphysical results. In the case of weak discontinuities, as discussed later, the troubles may be considered as apparent while in the case of shock waves, the situation does not appear to be so. All the studies that have been found thus far in shocks and most of the studies in weak discontinuities, with the exception of Juneja and Nariboli [37] and Seth [38], use the material system; all studies of the former [13-23] and most studies of the latter have been one-dimensional. The purpose of the discussion that follows will be to explain the consistency of these and bring out the basic point that, for any arbitrary three-dimensional shock wave travelling in an initially strained medium, use

of the material system⁵ leads to unsatisfactory results.

The basic difficulty involved in the use of a material system for shocks is the ambiguity involved in the definitions of various quantities associated with the shock surface. The weak discontinuity in the study of acceleration waves requires the strains to be continuous. Thus, the displacement gradients being continuous, a one-to-one correspondence exists between the spatial and material description. In the case of a shock wave, the strains themselves are discontinuous and thus the displacement gradients are discontinuous; hence such a correspondence is lost. A crude but illuminating way of restating this may be to assert that the shock surface (front) is characterized by the property that "two particles occupy the same spatial position". An elementary pictorial proof of this assertion can be inferred by a simple example given in most texts on fluid mechanics [39, p. 424] (Figure 2.2). The usual argument is as follows: Consider a reference wave profile of the form ABCDE, as

⁵Material system, undeformed system, and Lagrangian system are used interchangeably to mean the same.

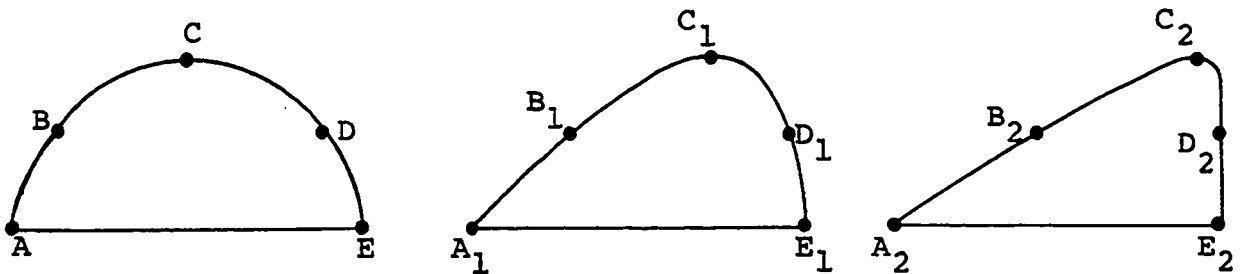


Figure 2.2. Pictorial representation of the formation of a shock wave

The gas-dynamical equations then assert that the speed at A is less than that at B while that at D is greater than that at E. Thus, the effect of nonlinearity of the basic equations is to flatten part ABC of the profile and to steepen part CDE as shown in the center diagram of Figure 2.2. With elapse of time, this tendency increases and ultimately a part of the profile assumes the vertical shape as in the last diagram of Figure 2.2. Thus at this stage, when the smooth solution breaks down and the shock is said to be formed, there are distinctly different elements D_2 and E_2 occupying the same place given by the abscissa of E_2 .

It is to be stressed that wave motion is not to be confused with the motion of particles. Particles in a wave do move but it is not the same as the motion of the wave. The motion of the wave is the motion of the pattern; at different instants it involves different particles; viz., in Figure 2.2 the particles at ABCDE do not move to $A_2B_2C_2D_2E_2$; the latter consists of different elements forming the corresponding pattern. Also the elements D_2 and E_2 which occupy the same place in the last diagram of Figure 2.2 had different identities before the wave arrived there. A more rigorous elaboration on this viewpoint is given by Truesdell [28, 29]. It is presently intended to bring out these points in a clear form.

The word particle, when used in continuum mechanics, has quite a different meaning; it simply is an identifiable element. It does not mean a chunk of matter in space surrounded by other chunks. There are no gaps in a continuum and thus the conventional notion of a particle is not valid for a continuum.

True, matter is known to be made up of particles and if one knows what happens to all the particles, one does know everything that needs to be known. But matter in aggregate possesses certain properties, which is all that may be of interest in the study of bulk matter. Thus here, as in all of continuum mechanics, though the word particle

is used, it conveys only the meaning of an identifiable element and nothing more.

Studies of a continuum are done in one of two ways, the spatial system and the material system. In the spatial view, attention is concentrated at points of space. Different particles come and leave the place; the interest is not in what happens to those individual elements but on what happens at that particular place. The entire picture is obtained by such a knowledge of all locations.

In the material view, one follows an individual element, say by coloring it (which it is assumed, does not affect the properties under investigation). This element moves to various places. Knowing about all such elements provides the description of the whole field.

To focus upon the apparent ambiguities of the material description of a shock wave, the material formulation of a singular surface is now given. Throughout this entire study a common fixed frame of reference is used. Also, rectangular cartesian coordinates are used. In this common frame, spatial points are denoted by x_i ($i,j,k,\dots=1,2,3$) and material points are denoted by X_A ($A,B,C,\dots=1,2,3$). Material coordinates are just labels or names of individual elements. Therefore there is no loss of generality in identifying material coordinates as initial positions (at $t=0$ or $t=t_0$ in general). Motion is then, in general, described by

$$x_i = x_i(X_A, t) ; \quad X_A = X_A(x_i, t) ;$$

$$\det(x_{i,A}), \det(X_{A,i}) \neq 0, \infty . \quad (2.4.1)$$

In order that the material does not separate one assumes the x_i 's are continuous functions of X_A . In general one can represent the singular surface $\Sigma(t)$ in terms of material coordinates as

$$F(X_A, t) = 0; \quad F(X_A, t) = f(x_i(X_A, t), t) . \quad (2.4.2)$$

At a certain fixed time t , $F=0$ represents the locus of the material points in the initial configuration, reached at the moment t by the shock front which at the same time has the position $f=0$ in the spatial system. (Cristescu [40, p. 443] and Truesdell [28, 29]).

Some aspects of the foregoing theory (representation of a shock wave by singular surface theory), while not losing their validity, lose their intuitive appeal when applied to the material variables. The shape of $\Sigma_0(t)$ (represented by $F=0$), including its first and second fundamental forms and its unit normal, has no immediate interpretation, for they do not correspond to any geometrical properties that an observer of a singular surface in space would perceive. (Truesdell [28, p. 507]).

Continuing:

The material representation, rather, is of the nature of a diagram for the moving surface. It is only one of many such diagrams, for by choice of the initial instant, or of the coordinates or parameters X_A corresponding to the given initial positions, the particular functional form that results from $F=0$ will differ.

As was mentioned earlier, all the known studies of shock waves [13-23] use the material system. And it is noted

that the source from which these studies draw is Truesdell [28], which is also the source of the above quotations.

In general, when one has a unique material representation for the singular surface, one can define a unit normal and normal speed; as in Equation 2.9.2, one has

$$F_{,t} + DP = 0; \quad P_A = F_{,A} = PN_A; \quad P_A P_A = P^2. \quad (2.4.3)$$

where N_A is the unit normal to $F=0$ and D is the normal speed of propagation in the material system. Note: Different words are used by different authors for D and G (given earlier) called here as normal speeds of propagation of the shock wave in the material and spatial systems respectively.

For a singular surface that is not a shock, it should be possible to relate uniquely all the geometrical quantities, some of which are found in the literature [28, p. 508]. It is the object of the remaining part of this section to show that, even for shocks, D (and also N_A) are continuous.

Now the definition of an elastic shock implies the discontinuity of $x_{i,A}$ which implies the discontinuity of strain. It can be asserted that the continuity of strain and velocity imply each other for a propagating singular surface [28, p. 519].

Now, the discontinuity of strain, stated by $[x_{i,A}] \neq 0$, implies that, to a single spatial surface $f(x_i, t) = 0$, there

corresponds two distinct diagrams (material representations), $F^+=0$ and $F^-=0$, viz.,

$$0 = f(x_i, t) = F^+(X_A^+(x_i, t), t) = F^-(X_A^-(x_i, t), t), \quad (2.4.4)$$

where the functions $X_A^+(x_i, t)$ and $X_A^-(x_i, t)$ are two inverse functions to the single-valued equation $x_i = x_i(X_A, t)$ [28, p. 513].

Thus for a shock wave one has two normal speeds of propagation D^\pm , two normal vectors N_A^\pm , etc. A prerequisite for validation of the use of a material description is to assert a relation or equality of these kinematical and geometrical quantities associated with a wave. Indeed continuity of the unit normal and the normal speed is known (though scarcely noted in the literature, further, the recognition of such a need also seems to be lacking).

To assert the continuity of D , first consider a tangential element (dx_A, dt) of the shock front in the material system. Referring to Figure 2.3, it is clear that for this element, since it is common to $F^+ = F^- = 0$, $dx_A^+ = dx_A^-$ and $dt^+ = dt^-$. This follows from the continuity of x_i as functions of X_A . For such a tangential displacement, using the continuity of x_i and Hadamard's lemma, one has [40, p. 444].

$$[x_{i,A}] dx_A + [x_{i,t}] dt = 0. \quad (2.4.5)$$

Further since (dx_A, dt) is a common element to $F^+=0$ and $F^-=0$, one has

$$F^+_{,A} dx_A + F^+_{,t} dt = 0, \quad (2.4.6)$$

$$F^-_{,A} dx_A + F^-_{,t} dt = 0. \quad (2.4.7)$$

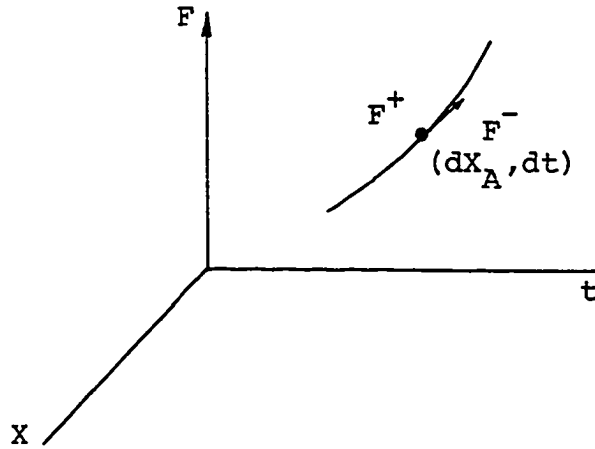


Figure 2.3. Material representation of the shock surface
(X_A is represented by the single axis X)

Since (dx_A, dt) is tangential to the singular surface, then from Equations 2.4.5, 2.4.6 and 2.4.7 it is concluded that $([x_{i,A}], [x_{i,t}])$, $(F^+_{,A}, F^+_{,t})$ and $(F^-_{,A}, F^-_{,t})$ all are orthogonal to this element and hence parallel to each other, giving

$$[x_{i,A}] = \lambda_i^+ F^+_{,A} = \lambda_i^- F^-_{,A}, \quad (2.4.8)$$

$$[x_{i,t}] = \lambda_i^+ F_{,t}^+ = \lambda_i^- F_{,t}^- . \quad (2.4.9)$$

Using Equations 2.4.2 and 2.3.2, the following result is obtained

$$F_{,A} = f_{,i} x_{i,A} = p n_i x_{i,A} ; \quad F_{,t} = \bar{f}_{,i} x_{i,t} + f_{,t} , \quad (2.4.10)$$

for both F^+ and F^- . Further noting that $[x_{i,A}] \neq 0$, one obtains

$$F_{,A}^+ - F_{,A}^- = p n_i [x_{i,A}] = p n_i \lambda_i^- F_{,A}^- = p n_i \lambda_i^+ F_{,A}^+ , \quad (2.4.11)$$

$$F_{,t}^+ - F_{,t}^- = p n_i [x_{i,t}] = p n_i \lambda_i^- F_{,t}^- = p n_i \lambda_i^+ F_{,t}^+ , \quad (2.4.12)$$

thus

$$F_{,A}^+ = (1 + p n_i \lambda_i^-) F_{,A}^- , \quad (2.4.13)$$

$$F_{,t}^+ = (1 + p n_i \lambda_i^-) F_{,t}^- . \quad (2.4.14)$$

Combining Equations 2.4.3, 2.4.13 and 2.4.14, the following result is obtained

$$D^+ = - \frac{F_{,t}^+}{P^+} = - \frac{(1 + p n_i \lambda_i^-) F_{,t}^-}{(1 + p n_i \lambda_i^-) P^-} = - \frac{F_{,t}^-}{P^-} = D^- , \quad (2.4.15)$$

where it was used that

$$\begin{aligned} P^+ &= (F_{,A}^+ F_{,A}^+)^{1/2} = (1 + p n_i \lambda_i^-) (F_{,A}^- F_{,A}^-)^{1/2} \\ &= (1 + p n_i \lambda_i^-) P^- , \end{aligned} \quad (2.4.16)$$

and

$$N_A^+ N_A^+ = N_A^- N_A^- = 1. \quad (2.4.17)$$

It is felt that this result is necessary in order to justify all the work done thus far using a material system and it does not appear to have been stressed well enough. In writing the shock conditions, which are derived later, the only quantities that need to be defined uniquely are the speed and the normal vector. The above arguments show the continuity of the normal vector too. Studies thus far have been one-dimensional. Since the shock-front is taken as planar, the normal vector can be chosen to be $(1,0,0)$ and the above results are sufficient to justify their results. However, curved shock waves in a strained elastic medium do critically depend, not only on the normal vector, but even on the curvatures of the shock-front which appear in the expression for the strains. Thus, unless one establishes the continuity of these too, one cannot use the material description for arbitrarily curved shocks in an initially strained elastic medium. (For the initially unstrained case curvatures do not appear, at least in the shock conditions.) All these considerations are needed for a study of shocks in a material system. And even after these results are established, it does remain physically obscure unless transformed to physical space. However, it is not the

intention of this study to establish these results, since, in view of earlier remarks, all the results of the studies in the material system acquire physical import only after they are transformed to the spatial system. So in the forthcoming study a complete spatial formulation is sought.

In order to further emphasize why curvatures of the wave-front may be important in general, consider the general equation governing the growth⁶ of a weak discontinuity in acceleration waves (also called second-order waves, weak waves and sonic waves by various authors); for a number of problems, it is reducible to the form [19, 25, 30, 41]

$$\frac{d\psi}{dt} + A\psi + B\psi^2 + C\psi = 0 , \quad (2.4.18)$$

where d/dt is the ray derivative.

Here terms can be interpreted as follows: ψ is the strength of the discontinuity, defined in a suitable way (e.g. in this study it depends on the jump in the second normal derivative of the displacement vector; A gives the effect of curvature of the wave-front; B is the term reflecting the nonlinearity and drops out in a linearized problem; but is the most crucial term governing the growth of the wave and therefore shock formation; the last term C is only present when the system is dissipative but yet still allows the basic field equations to be hyperbolic, e.g., a

⁶Nariboli, G. A. Ames, Iowa. Class notes. Private communication. 1969.

heat-conducting inviscid gas [42, 43] and materials with memory [19]. When C is present, it shows exponential decay of the wave. The crucial difficulty in using the material system is with regard to the curvature term. The asymptotic structure of the wave critically depends on this term [44] and indicates regions where the disturbance is expected to be strongest, which is indeed important in the case of observations. Thus, what is needed is the spatial system where the disturbance is strongest.

For the case when strains are continuous, corresponding to each compatibility condition in the spatial system one can write the dual relation in the material system. Thus working with the material system continues to be completely valid here, though it is felt that it will be physically meaningful only if transformed to the spatial system. In the case of shock waves, such a duality or at least the correspondence must be established.

The above discussion can appear to be quite controversial since use of the material system has been quite common. These arguments are provided to assert the feasibility of working with the spatial system. However, whatever ones views are with regard to the above objections, the subsequent formulation in the spatial system is not without novelties.

2.5. Shock Conditions

There appear to be two approaches in deriving the shock conditions. The first one that is presented here is common in the literature on continuum mechanics. The second one is common to the literature in partial differential equations.

In the first approach basic equations are written in an integral form. In regions where derivatives of field variables exist, one can obtain the differential forms. In regions where the field variables suffer discontinuities across certain surfaces, the integral forms yield shock conditions (or jump conditions).

The basic equations of a continuum are those of conservation of mass (or the equation of continuity), balance of linear momentum, balance of angular momentum and balance of energy. These lead to what are called conservative forms of partial differential equations [45]. First one considers the Reynold's transport theorem [28, p. 347].

$$\frac{d}{dt} \int_{V(t)} \psi dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{S(t)} \psi v_n dS . \quad (2.5.1)$$

Here d/dt is the material derivative, $V(t)$ is an arbitrary moving volume and $v_n = v_i n_i$ is the normal speed of the boundary of $V(t)$, denoted by $S(t)$, unit normal to which

is n_i . Further ψ is any function, which may be a tensor.

Consider now a discontinuity surface $\Sigma(t)$ (the singular surface) across which ψ may suffer a jump. Surround $\Sigma(t)$ by a volume $V(t)$ which is divided by $\Sigma(t)$ into two parts $V_1(t)$ and $V_2(t)$.

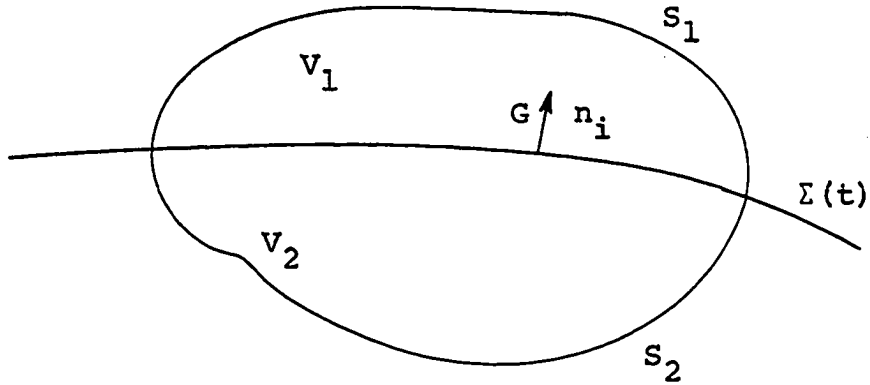


Figure 2.4. Singular surface moving in an arbitrary volume

Clearly the boundary of V_1 (or V_0) consists of S_1 , which is a part of the boundary of V and $\Sigma(t)$, while that of V_2 consists of S_2 and $\Sigma(t)$. Let positive n_i be taken to point into V_1 for convenience. In the present study the following convention is adopted: the jump (denoted by a bracket) denotes the value of the field variable in the shocked medium (behind) minus the value of the same in the unshocked medium (ahead). Further the positive normal n_i always points into the unshocked region and the normal speed G of the singular surface is positive in the direction

of this normal. It is to be remembered that shock conditions remain unchanged as long as the same order is maintained.

Now one writes

$$\frac{d}{dt} \int_{V(t)} \psi dV = \frac{d}{dt} \int_{V_1(t)} \psi dV + \frac{d}{dt} \int_{V_2(t)} \psi dV . \quad (2.5.2)$$

Apply then the transport theorem separately to V_1 and V_2 . However, v_n of Equation 2.5.1 has to be identified with the normal speed of the medium on the boundary of S_1 and S_2 and with that of $S(t)$ on $\Sigma(t)$. Thus for V_2 , one has

$$\frac{d}{dt} \int_{V_2(t)} \psi dV = \int_{V_2} \frac{\partial \psi}{\partial t} dV + \int_{S_2(t)} \psi_2 (-v_{2n}) dS_2 + \int_{\Sigma(t)} \psi_2 G d\Sigma . \quad (2.5.3)$$

Here one understands by ψ_2 in the first surface integral as the value on S in the region V_2 evaluated on the boundary S_2 , while ψ_2 in the last integral means the value of ψ in the region V_2 evaluated on the boundary Σ .

Writing a similar relation for V_1 and adding, one obtains

$$\frac{d}{dt} \int_V \psi dV = \int_V \frac{\partial \psi}{\partial t} dV + \int_S \psi v_n dS + \int_{\Sigma} (\psi_2 - \psi_1) G d\Sigma . \quad (2.5.4)$$

Let now V shrink to zero at a fixed time t such that $S_1 \rightarrow \Sigma$ and $S_2 \rightarrow \Sigma$. Further assume $\partial \psi / \partial t$ remains bounded such

that the first integral vanishes in the limit. So one obtains

$$\begin{aligned} \frac{d}{dt} \int_V \psi dV + \int_{S_2} \psi_2 (-v_{2n}) dS + \int_{\Sigma} \psi_2 G d\Sigma \\ + \int_{S_1} \psi_1 (v_{1n}) dS + \int_{\Sigma} \psi_1 (-G) d\Sigma = - \int_{\Sigma} [(\mathbf{v}_n - G)\psi] d\Sigma. \end{aligned} \quad (2.5.5)$$

It is to be remembered that in the limit as stated above, v_n for V_2 is negative while for V_1 it is positive and similar considerations hold for G . This follows from the convention that, in the divergence theorem, the exterior normal is taken as the positive normal.

Now the basic laws of balance for a continuum can all be written in the form

$$\frac{d}{dt} \int_V \psi \dots dV = \int_S \phi \dots n_i dS + \int_V \chi \dots dV. \quad (2.5.6)$$

Here dots denote indices so that the above equation is tensorially balanced. Use of Equation 2.5.5 with Equation 2.5.6, yields

$$[(\mathbf{v}_n - G)\psi \dots] = [\phi \dots] n_i. \quad (2.5.7)$$

In arriving at the above, it is firstly noted that the volume integral does not contribute and further n_i has to be taken with opposite signs for the two regions.

The basic laws of balance for a non-polar and non-heat

conducting continuum (with negligible or zero magnetic and electrical energy) are taken as

$$\frac{d}{dt} \int_V \rho dV = 0, \quad (2.5.8)$$

$$\frac{d}{dt} \int_V \rho v_i dV = \int_S t_{ji} n_j dS + \int_V \rho f_i dV, \quad (2.5.9)$$

$$\frac{d}{dt} \int_V \epsilon_{ijk} \rho x_j v_k dV = \int_S \epsilon_{ijk} x_j t_{lk} n_l dS + \int_V \epsilon_{ijk} \rho x_j f_k dV, \quad (2.5.10)$$

$$\frac{d}{dt} \int_V \rho (e + \frac{1}{2} v_i v_i) dV = \int_S t_{ji} n_j v_i dS + \int_V \rho v_i f_i dV. \quad (2.5.11)$$

Here e is the internal energy per unit mass, f_i is the body force vector per unit mass, t_{ji} is the Eulerian (spatial) stress tensor per unit deformed area, ϵ_{ijk} is the conventional alternating or permutation tensor and v_i is the velocity vector. These lead to differential forms in regions of validity; viz.,

$$\frac{d\rho}{dt} + \rho v_{i,i} = 0, \quad (2.5.12)$$

$$\rho \frac{dv_i}{dt} = t_{ji,j} + \rho f_i, \quad (2.5.13)$$

$$\rho \frac{de}{dt} = t_{ji} v_{i,j} = t_{ji} d_{ij}, \quad (2.5.14)$$

where

$$2d_{ij} = v_{i,j} + v_{j,i} , \quad (2.5.15)$$

and shock (jump) conditions across shock waves as

$$[\rho(v_n - G)] = 0; \quad v_n = v_i n_i , \quad (2.5.16)$$

$$[\rho(v_n - G)v_i] = [t_{ji}n_j] = [t_{ji}]n_j , \quad (2.5.17)$$

$$[\rho(v_n - G)(e + \frac{1}{2} v_i v_i)] = [t_{ji}n_j v_i] = [t_{ji}v_i]n_j , \quad (2.5.18)$$

where the following defined bracket operations are used above and throughout this study:

$$[ab] = a_1 b_1 - a_0 b_0 , \quad (2.5.19)$$

$$[ab] = [a][b] + a_0 [b] + b_0 [a] , \quad (2.5.20)$$

$$[ab] = -[a][b] + a_1 [b] + b_1 [a] , \quad (2.5.21)$$

$$[ab] = \bar{a}[b] + \bar{b}[a]; \quad \bar{d} = (d_1 + d_2)/2 . \quad (2.5.22)$$

Equation 2.5.10, which expresses the balance of moment of linear momentum, in differential form only leads to

$$\epsilon_{ijk} t_{ji} = 0 , \quad (2.5.23)$$

expressing the symmetry of the stress tensor. The corresponding form of Equation 2.5.10 for jump conditions across the shock wave leads simply to an identity and thus no new condition is obtained.

Consider now the second viewpoint. The basic differential system for the balance laws for a continuum given in Equations 2.5.12, 2.5.13 and 2.5.14 can be written in a conservative form as [45]

$$\frac{\partial \psi_{...}}{\partial t} + \frac{\partial \phi_{...}}{\partial x_i} = \chi_{...} , \quad (2.5.24)$$

where the dots again denote tensorial indices so that the above equation is tensorially of the correct form.

The equation of continuity, Equation 2.5.12, is put in the conservative form by just expanding the material derivative as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0 . \quad (2.5.25)$$

Multiply the continuity equation by v_i and add it to the equations of motion, given in Equation 2.5.13, to obtain

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j - t_{ji}) = \rho f_i . \quad (2.5.26)$$

Lastly, multiply the equation of continuity by $e + \frac{1}{2} v_i v_i$, the equations of motion by v_i and add these to the energy equation, given in Equation 2.5.14, to obtain the final equation in the conservative form, as

$$\frac{\partial}{\partial t} \left\{ \rho \left(e + \frac{1}{2} v_i v_i \right) \right\} + \frac{\partial}{\partial x_j} \left\{ \rho v_j \left(e + \frac{1}{2} v_i v_i \right) - t_{ji} v_i \right\} = \rho f_i v_i . \quad (2.5.27)$$

Now, Equations 2.5.25, 2.5.26 and 2.5.27 are of the needed form given in Equation 2.5.24. The concept of a weak solution generalizes the notion of the usual solutions so as to also include shock solutions as possible solutions. These ideas are rigorously based on the theory of distributions for which extensive treatments exist [31, 45, 46]. Only the physical content of these is presented here.

When field variables are differentiable, the basic equations do have meaning. However, when this fails the operation of differentiation loses meaning. Still in order to define solutions of the basic equations one introduces a test function g which possesses continuous derivatives as many times as needed and vanishes on the boundary of the domain of interest. The construction of such functions is discussed in relevant literature [47]. Assuming this, one considers a region R , on the boundary of which g vanishes. Multiply Equation 2.5.24 by g and then integrate each term in Equation 2.5.24 separately over the region. Then one has

$$\int_{\tau} g \frac{\partial \psi}{\partial t} d\tau = \int_{\tau} \frac{\partial}{\partial t} (g\psi) d\tau - \int_{\tau} \psi \frac{\partial g}{\partial t} d\tau = - \int_{\tau} \psi \frac{\partial g}{\partial t} d\tau, \quad (2.5.28)$$

since g vanishes on the boundary.

Similarly one has

$$\int_{\tau} g \frac{\partial \phi}{\partial x_i} d\tau = - \int_{\tau} \phi \frac{\partial g}{\partial x_i} d\tau. \quad (2.5.29)$$

Here τ is the four-dimensional region of (x_i, t) .

Equation 2.5.24 is now written as

$$\int_{\tau} (\psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + \chi g) d\tau = 0. \quad (2.5.30)$$

The dot suffixes on ϕ , ψ and χ are suppressed above and throughout the rest of this section for convenience; but are to be implicitly understood.

Now Equation 2.5.30 defines the weak solution of Equation 2.5.24 and is taken as the starting point. If ψ and ϕ are differentiable, carrying out the above steps in the reverse order, one recovers

$$\int_{\tau} g \{ \frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial x_i} - \chi \} d\tau = 0. \quad (2.5.31)$$

Since this is true for an arbitrary g satisfying the invoked requirements, the bracketed terms must vanish, recovering the original equation [31, 46, 48-50].

Consider again Figure 2.4 where region V (now four-dimensional); bounded by $S_1 + S_2$ is divided into two regions V_1 and V_2 by $\Sigma(t)$. Form first the identity

$$\frac{\partial (g\psi)}{\partial t} + \frac{\partial (g\phi)}{\partial x_i} = (\psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + g\chi) + g(\frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial x_i} - \chi). \quad (2.5.32)$$

Assume Equation 2.5.24 is satisfied in each of V_1 and V_2 separately. Integrating Equation 2.5.32 in V_2 (now the last bracketed terms vanish since g vanishes on the boundary and the left members of Equation 2.5.32 are transformed

by use of the divergence theorem), one has

$$\begin{aligned} & \int_{S_2} (n_4 g \psi + n_i g \phi) dS_2 + \int_{\Sigma(t)} (n_4 g \psi + n_i g \phi) d\Sigma \\ &= \int_{V_2} \left\{ \psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + g \chi \right\} d\tau, \end{aligned} \quad (2.5.33)$$

where S_2 is the boundary of the four-dimensional region V_2 , S_1 is the boundary of the four-dimensional region V_1 , such that $S_1 + S_2 = S$. $\Sigma(t)$ is of course the singular surface and (n_i, n_4) are the components of the normal to $\Sigma(t)$ in the four-space.

To obtain this unit four-normal vector consider a Euclidean four-space with rectangular coordinates $x_{i'}$, ($i' = 1, 2, 3, 4$) where $x_4 = t$ for $i' = 4$ and $x_{i'} = x_i$ for $i' = i = 1, 2, 3$. Then $\Sigma(t)$ is represented by [36]

$$f(x_{i'}) = 0. \quad (2.5.34)$$

Then the normal to $\Sigma(t)$ is given by

$$p_{i'} = p' n_{i'}, \quad (2.5.35)$$

where

$$p_{i'} = f_{,i'}; \quad p'^2 = p_4^2 + p_i p_i = p_4^2 + p^2. \quad (2.5.36)$$

However, from Equation 2.3.4, $p_4 + Gp = 0$ and $p_i = p n_i$, which combined with Equation 2.4.36 yields

$$p'^2 = (1 + G^2) p^2, \quad (2.5.37)$$

$$n_{i'} = \frac{1}{\sqrt{1+G^2}p^2} (n_i, -G) , \quad (2.5.38)$$

$$n_i, n_{i'} = 1. \quad (2.5.39)$$

Now the first integral on the left hand side of Equation 2.5.33 vanishes since g vanishes on S_2 . Note further that the functions in the integrand in the surface integrals must be evaluated on the surface. Thus, bringing this out, one has

$$\int_{\Sigma(t)} (n_4 g_2 \psi_2 + n_i g_2 \phi_2) d\Sigma = \int_{V_2} \left\{ \psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + g\chi \right\} d\tau. \quad (2.5.40)$$

Noting that for V_1 , the normal vector is $(-n_i, -n_4)$ (following the convention of Figure 2.4), one similarly obtains for V_1

$$\int_{\Sigma(t)} (-n_4 g_1 \psi_1 - n_i g_1 \phi_1) d\Sigma = \int_{V_1} \left\{ \psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + g\chi \right\} d\tau. \quad (2.5.41)$$

Now g is continuous across $\Sigma(t)$, so the addition of Equations 2.5.40 and 2.5.41 gives

$$\int_{\Sigma(t)} g \{ n_4 (\psi_2 - \psi_1) + n_i (\phi_2 - \phi_1) \} d\Sigma = \int_V \left\{ \psi \frac{\partial g}{\partial t} + \phi \frac{\partial g}{\partial x_i} + g\chi \right\} d\tau. \quad (2.5.42)$$

The right hand side of Equation 2.5.42 defines the weak solution and hence it vanishes. Further the left hand side

is for any arbitrary portion of $\Sigma(t)$ and so the integrand vanishes, yielding

$$n_4[\psi] + n_i[\phi] = 0. \quad (2.5.43)$$

Noting that

$$n_4 = \frac{-G}{\sqrt{1+G^2}} ; n_i \rightarrow \frac{n_i}{\sqrt{1+G^2}}$$

from Equation 2.5.38, where $n_{i,} = (n_i, n_4)$ is a unit four-normal vector and n_i is the unit three-normal vector. Equation 2.5.43 reduces to

$$-G[\psi] + n_{i,}[\phi] = 0. \quad (2.5.44)$$

To establish the relationship between these two approaches for obtaining the jump conditions, one first notes that

$$\frac{d}{dt} \int_V \psi dV = \int_V \left\{ \frac{\partial \psi}{\partial t} + (\psi v_i)_{,i} \right\} dV. \quad (2.5.45)$$

Thus one can write the conservative form of the basic equations, given in Equation 2.5.24, as

$$\frac{\partial \psi}{\partial t} + (\psi v_i)_{,i} + \frac{\partial}{\partial x_i} (\phi - \psi v_i) = \chi. \quad (2.5.46)$$

Using the first view developed, i.e., singular surface theory, Equation 2.5.46 leads to

$$[(v_n - G)\psi] + n_i[\phi - \psi v_i] = 0, \quad (2.5.47)$$

which simplifies to the same as Equation 2.5.44. Thus the relationship is established.

Two points are stressed at the cost of repetition. The indices on ψ , ϕ and χ were omitted in Equations 2.5.28-2.5.47 and so when provided, these equations do remain tensor equations. Further geometrical quantities, e.g., n_i , can be taken in or out of the square brackets according to needs; in the spatial system their continuity is obvious. The technique of using the conservative form does not make it obvious that, in the material system, the defined normal vector, normal speed, etc. have to be proved to be continuous. Thus one can formally escape the need of proving it. It is only the physical view of a material four-volume V , in a problem involving a lack of one-to-one correspondence in spatial to material transformation, which brings out this need.

2.6. Admissibility Criteria for Shock Waves

The theory of weak solutions does not define weak solutions uniquely. There is an indeterminacy. One can see that there is at least one additional unknown, the shock speed. For plane shocks moving with constant speed, this can formally be removed by a Galilean transformation, valid

for Newtonian mechanics which is the framework of this study. However, even then, an additional quantity has to be specified. It is the author's experience that it is not very well recognized in the literature that this transformation holds only for plane shock waves with constant speed. A number of criteria discussed below are only true for plane shocks. But shocks need not be planar; a criteria must hold for all shocks. For curved shocks, where constant speed of propagation is not a valid assumption, few of these criteria can be applied. It appears to be still open whether these criteria are necessary or sufficient too. The necessity appears to be obvious from physical considerations; but whether these suffice to determine the unique solution does not appear to be clear. Moreover, since some of these criteria are not valid for curved or non-uniform shocks, it appears that at the present stage physical considerations do still remain the decisive factor in choosing the correct solution.

A short review of these methods now follows:

1. Viscosity method: Shock solutions are solutions in an ideal material. In real materials these ideal discontinuity surfaces are extremely narrow transition-regions. The state of the medium changes from one to the other in distances extremely small compared to other relevant distances in the problem. In gas-

dynamics it is well known to be determined by the magnitude of the viscosity and turns out to be of the order of the mean free path of constituent molecules [1, p. 137]. Hence objections can be raised to the study of shock waves from a continuum viewpoint. But it appears to be accepted that the study is still extremely useful for all practical purposes. In this method, which is also known as the study of the structure of shock waves, one introduces viscosity and a Galilean frame into the governing one-dimensional equations.

This method automatically decides not only the admissibility but also the initial (unshocked) and the final (shocked) states. It appears that the difficulty in the use of this method is that a fairly adequate knowledge of state functions is necessary.

There are however two limitations in the use of this method. The shock must be one-dimensional and steady in a Galilean frame. Thus for curved shocks where no such Galilean frame can exist and for unsteady one-dimensional shocks, the method is difficult to apply.

2. Evolutionary condition: This is very clearly stated in the book by Jeffrey and Taniuti [30, p. 124]. This is again limited to one-dimensional steady (constant

shock speed) shock waves. So it suffices to reproduce the result.

Let⁷ the governing one-dimensional equations be represented by

$$\underline{u}_t + \underline{f}_x = 0; \quad \underline{f} = \underline{f}(\underline{u}) ,$$

or

$$\underline{u}_t + A\underline{u}_x = 0; \quad A = \partial \underline{f} / \partial \underline{u} , \quad (2.6.1)$$

where \underline{u} is a vector of field variables, \underline{f} is a vector function of \underline{u} . The condition now is stated as

A discontinuity is evolutionary if and only if the number of small amplitude outgoing waves diverging from the discontinuity is equal to the number of the boundary conditions minus one, and at the same time the eigenvectors of A , corresponding to these outgoing waves, and the vector \underline{u} are linearly independent provided of course that the disturbed boundary conditions resulting from the shock conditions are independent.

3. Stability condition: This type of stability analysis is best illustrated by contributors in what is known as hydrodynamic stability. Here one linearizes the basic equations about a known state and the perturbed system leads to an eigenvalue problem. Solution of such an eigenvalue problem gives a criterion of stability; viz., as to whether the small perturbations

⁷An underscore (~) denotes vectors, tensors, etc.

grow or die down with time.

Analogously since the concept of weak solutions does not determine unique solutions, it is possible to invoke that only those solutions which are stable against small time-dependent perturbations should be regarded as admissible ones. Such studies of applying hydrodynamic stability analysis to the theory of weak solutions do exist in recent literature [51-55]. Studies with nonlinear terms retained do not exist, although this question has been raised [52].

However these studies are limited to one-dimensional steady shocks. Thus for curved expanding shocks, which are not steady, this method is not applicable.

4. Entropy condition: This has been the most commonly known condition in gas-dynamics. Shock conditions do admit both compressive and rarefaction shocks; physically one knows only compressive shocks exist; the entropy condition picks out only the compressive shock as admissible.

This condition is invoked simply from the basic principle that in any process entropy cannot decrease.

So across a shock one invokes that entropy must increase. Sonic waves are found to be isentropic and weak shocks in the limit can be identified as sonic waves.

Such a constraint produces an inequality for the relation between state variables. In gas-dynamics this states that $P_{\tau\tau} \big|_0 [\tau]^3 < 0$ where P is the pressure, τ is the reciprocal of density and the square bracket denotes the jump in τ . For a compressive shock $[\tau]$ is negative and hence $P_{\tau\tau} \big|_0$, the second partial derivative of p with respect to τ with entropy held constant (denoted by the suffix "o"), must be positive. If there exist materials for which $P_{\tau\tau} \big|_0$ is negative, then of course the rarefaction shocks are the ones that occur in these materials.

Such equations are thus exceedingly useful in producing constraints on the material properties. The present study does produce some such inequalities. However in gas-dynamics, the results obtained are proved more generally. The above argument for $P_{\tau\tau} \big|_0$ implies the validity of a Taylor series expansion and the small-ness of $[\tau]$ allows the neglect of higher powers; the more general proof discards this requirement. It has not been

possible to produce such a general inequality in the present study, though, in particular cases, from analogy with gas-dynamics, when it holds, it does seem possible.

5. Growth of weak waves: This observation appears to be stated here for the first time.⁸ The general equation governing the growth of a weak discontinuity, which is a propagating singular surface (wave) whose normal speed of propagation depends on the state ahead only and not on the amplitude of the wave, can be written as

$$\frac{d\psi}{dt} + B\psi^2 + (A+C)\psi = 0, \quad (2.6.2)$$

where the symbols are the same as in Equation 2.4.18.

Here the crucial term determining the growth is B. It is the sign of this term that determines whether a wave grows or not; i.e., whether a wave will grow into a shock wave or not.

Further from the study of simple waves, one can obtain the critical time when characteristics intersect and form an envelope. This critical time interpreted as the time of shock formation has to be positive [56].

⁸Nariboli, G. A. Ames, Iowa. Class notes. Private communication. 1969.

One also obtains via the entropy condition an analogous condition to the one obtained through Equation 2.4.18, which is obtained from singular surface theory.

It is asserted that one can verify that for gas-dynamics all three of the above, singular surface theory, simple waves and the entropy condition, involve the positivity of the same quantity

Thus, this new view may provide another criterion. Also this criterion may be called the condition imposed from the requirement that an admissible shock is formed from a smooth solution.

Though in certain cases these five criteria lead to mutually consistent or even identical predictions, an exhaustive study to relate all these appears to be lacking. See however Lynn [55] for some discussion.

3. FORMULATION OF THE ELASTIC PROBLEM

The object of this chapter is to collect together fairly well known results used in the study of wave propagation. However, the final formulation of the constitutive law for a hyperelastic body⁹ based on the deformed (spatial) configuration is believed to be new. For the isotropic case, this interesting formulation places the less known Hencky strain measure as preferable to others; its introduction is required only by thermodynamic considerations.

As before, a common fixed frame x_i and X_A are used to denote spatial and material coordinates respectively, both are taken as rectangular, cartesian coordinates throughout. Since spatial and material variables are used together in this chapter, as in section 2.4, small letters with latin subscripts are used for spatial variables and capital letters with capital latin subscripts are used for material variables: This will be strictly followed throughout this chapter unless explicitly written otherwise. Further cartesian tensor notation is used. Still further occasionally use is made of letters with an underscore to denote vectors and tensors; their tensorial character and their reference system (deformed or undeformed) have all to be understood from context only.

⁹Body, medium and material are used interchangeably to mean the same.

3.1. Strain Tensors

There exists no uniform notation nor use of any one strain tensor by authors in the field; even the names attributed to each do not appear to be common. Only the strain tensors used in wave propagation problems will be noted here. Thus tensors in the convected system, used mostly by Green and Zerna [57], will be omitted for this reason. Some authors do not distinguish indices for the deformed and undeformed configuration [23, 58]; this distinction will be strictly followed in the present work.

The different strain tensors appearing in the literature are:

$$c_{ij} = X_{A,i} X_{A,j}, \quad \text{Cauchy [28, p. 257],} \quad (3.1.1)$$

$$C_{AB} = x_{i,A} x_{i,B}, \quad \text{Green [28, p. 257]; right Cauchy-Green [59, p. 53],} \quad (3.1.2)$$

$$2e_{ij} = \delta_{ij} - c_{ij}, \quad \text{Almansi-Hamel [28, p. 266],} \quad (3.1.3)$$

$$2E_{AB} = C_{AB} - \delta_{AB}, \quad \text{Green-St. Venant [28, p. 266],} \quad (3.1.4)$$

$$(c^{-1})_{ij} \equiv \text{reciprocal of } c_{ij} = x_{i,A} x_{j,A}, \quad \text{Finger [28, p. 263]; left Cauchy-Green [59, p. 53],} \quad (3.1.5)$$

$$(C^{-1})_{AB} \equiv \text{reciprocal of } C_{AB} = x_{A,i} x_{B,i}, \quad \text{Piola [28, p. 263].} \quad (3.1.6)$$

Further introducing the displacement vectors,

$u_i = u_i(x_i, t)$ or $U_A = U_A(X_A, t)$, one can write

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i} u_{k,j} \quad , \quad (3.1.7)$$

$$2E_{AB} = U_{A,B} + U_{B,A} - U_{D,A} U_{D,B} \quad . \quad (3.1.8)$$

It is noted that Equations 3.1.7 and 3.1.8 are the only tensorial strain expressions where the reference remains completely spatial or completely material respectively. The strain tensor in Equation 3.1.7 is often called the Eulerian strain tensor and the strain tensor in Equation 3.1.8 is often called the Lagrangian strain tensor. Thus for a completely spatial formulation, which is believed to be more appropriate for wave propagation in general, and for shocks more particularly, Equation 3.1.7 is the only appropriate one; other strain measures defined in terms of this tensor are also equally feasible. One regards u_i as the basic field variables defined as spatial variables and their spatial gradients define a strain. Use of this completely avoids any reference to the material system.

In the formulation that follows, one further needs the material derivatives of various strain tensors. Also needed are the following formulas which follow from the chain rule of differentiation

$$x_{i,A} x_{A,j} = \delta_{ij}; \quad x_{A,i} x_{i,B} = \delta_{AB} \quad . \quad (3.1.9)$$

If v_i is the spatial velocity vector, one has

$$v_i = \frac{dx_i}{dt} ; x_i = x_i(X_A, t) , \quad (3.1.10)$$

$$v_i = \left. \frac{\partial x_i}{\partial t} \right)_{X_A=\text{constant}} , \quad (3.1.11)$$

where X_A are substituted in terms of the x_i after differentiation.

Further one can write the spatial velocity vector in terms of the displacement vector, u_i , as

$$v_i = \left. \frac{du_i}{dt} \right)_{X_A=\text{constant}} , \quad (3.1.12)$$

where

$$u_i = u_i(x_i(X_A, t), t) , \quad (3.1.13)$$

or

$$v_i = \frac{\partial u_i}{\partial t} + u_{i,j} v_j . \quad (3.1.14)$$

Thus taking the material derivative of the first of Equation 3.1.9 yields

$$x_{A,j} \frac{d}{dt} (x_{i,A}) + x_{i,A} \frac{d}{dt} (x_{A,j}) = 0 , \quad (3.1.15)$$

$$v_{i,j} + x_{i,A} \frac{d}{dt} (x_{A,j}) = 0 , \quad (3.1.16)$$

or

$$v_{i,j} = -x_{i,A} \frac{d}{dt} (x_{A,j}) . \quad (3.1.17)$$

Multiplying Equation 3.1.17 by $(X_{C,i})$ and making use of the second of Equation 3.1.9 yields

$$\frac{d(X_{A,j})}{dt} = -X_{A,i}v_{i,j} \quad (3.1.18)$$

A relation similar to Equation 3.1.18 but with the displacement vector as the basic variable is also needed. If one expands the material derivative of $u_{i,j}$ and makes use of Equation 3.1.18, one obtains

$$\frac{du_{i,j}}{dt} = X_{A,j} \frac{d}{dt} (u_{i,A}) + u_{i,A} \frac{d}{dt} (X_{A,j}) \quad (3.1.19)$$

$$= v_{i,j} - u_{i,A} X_{A,k} v_{k,j} \quad (3.1.20)$$

or

$$\frac{du_{i,j}}{dt} = v_{i,j} - u_{i,k} v_{k,j} \quad (3.1.21)$$

After using Equations 3.1.18 and 3.1.21 with Equations 3.1.1-3.1.4 and 3.1.7, one can prove the following:

$$\frac{dC_{AB}}{dt} = 2d_{ij}x_{i,A}x_{j,B} \quad (3.1.22)$$

$$\frac{dc_{ij}}{dt} = -c_{ik}v_{k,j} - c_{jk}v_{k,i} , \quad (3.1.23)$$

$$\frac{dE_{AB}}{dt} = d_{ij}x_{i,A}x_{j,B} , \quad (3.1.24)$$

$$\frac{de_{ij}}{dt} = d_{ij} - (e_{ik}v_{k,j} + e_{jk}v_{k,i}) , \quad (3.1.25)$$

where

$$2d_{ij} = v_{i,j} + v_{j,i} . \quad (3.1.26)$$

These will suffice for the present purpose.

3.2. Stress Tensors

As with strain tensors, different stress tensors continue to be used in the study of problems of wave propagation. Recorded therefore are the three commonly used ones in wave propagation with only spatial, mixed spatial-material and only material indices.

One considers an imaginary surface s dividing the material in the current (deformed or spatial) configuration into two regions called again R_1 and R_2 . Further let n_i be the unit normal to s pointing into R_1 . Then one makes a hypothesis of a stress vector t_i such that the force per unit deformed area exerted by the medium in region R_1 on the region in R_2 across a surface element $ds_i = n_i ds$ containing n_i , is given by this vector t_i . Applying the equations of

motion for an elementary tetrahedron, one can prove it is possible to introduce a symmetric stress tensor as

$$t_i = t_{ji}n_j; \quad t_{ij}=t_{ji} . \quad (3.2.1)$$

To introduce the other stress tensors in use one needs a relation between elements of areas in the deformed and undeformed configurations. To this end consider first the element in the deformed configuration given by

$$ds_i = n_i ds = \epsilon_{ijk} dx_j \delta x_k , \quad (3.2.2)$$

where dx_j and δx_k are sides of the elementary parallelogram forming the sides of ds_i and ϵ_{ijk} is again the conventional permutation tensor.

The element of an area in the undeformed configuration bounded by sides dX_B and δX_C is given by

$$dS_{OA} = N_A dS_O = \epsilon_{ABC} dX_B \delta X_C . \quad (3.2.3)$$

If now one takes dX_B and δX_C transformed to dx_j and δx_k at a fixed time t , and using Equation 3.2.2, one obtains

$$ds_i = \epsilon_{ijk} x_{j,B} x_{k,C} dX_B dX_C . \quad (3.2.4)$$

Consider the expansion of the Jacobian of transformation

$$J = |x_{i,A}|; \quad \epsilon_{ABC} J = \epsilon_{ijk} x_{i,A} x_{j,B} x_{k,C} . \quad (3.2.5)$$

Multiply the second of Equation 3.2.5 by $X_{A,\ell}$ and using $x_{i,A}X_{A,\ell} = \delta_{i\ell}$, one obtains

$$\epsilon_{ABC} J X_{A,i} = \epsilon_{ijk} x_{j,B} x_{k,C} . \quad (3.2.6)$$

Thus use of Equation 3.2.6 in Equation 3.2.4 gives the result

$$ds_i = J X_{A,i} dS_{oA} . \quad (3.2.7)$$

One now introduces another stress vector T_A at a point x_i referred to the undeformed area dS_{oA} at $X_A = X_A(x_i, t)$ by

$$t_i ds_i = T_A dS_{oA} , \quad (3.2.8)$$

or

$$\begin{aligned} n_j t_{ji} ds_i &= n_j t_{ji} J X_{A,i} dS_{oA} \\ &= n_j T_{jA} dS_{oA} . \end{aligned} \quad (3.2.9)$$

This allows one to define the Piola (Kirchhoff) stress tensor [28, p. 553; 60, p. 109] as

$$T_{jA} = J X_{A,i} t_{ji} ; \quad t_{ji} = J^{-1} x_{i,A} T_{jA} . \quad (3.2.10)$$

Thus T_{iA} gives the stress at x_i measured per unit area at X_A . The quantity T_{iA} is the component along the X_A -coordinate, of the component of the stress vector along the x_i -coordinate, multiplied by the ratio of area at x_i to the area at X_A . Thus the quantities T_{iA} , sometimes called

pseudo-stresses, are awkward to interpret. The property of symmetry of the stress tensor, following from the balance of moment of momentum (called Cauchy's second law), assumes the form

$$T_{iA} x_{j,A} = T_{jA} x_{i,A} . \quad (3.2.11)$$

To transform the equation of balance of linear momentum (called Cauchy's first law), consider first the expansion of the Jacobian of transformation for X_A to x_i , as

$$j = J^{-1} = |X_{A,i}|; \quad \epsilon_{ABC} J^{-1} = \epsilon_{ijk} X_{A,i} X_{B,j} X_{C,k} . \quad (3.2.12)$$

Multiplying Equation 3.2.12 by $x_{\ell,A}$ and using $x_{\ell,A} X_{A,i} = \delta_{i\ell}$, one obtains, after renaming indices,

$$\epsilon_{ABC} x_{i,A} J^{-1} = \epsilon_{ijk} X_{B,j} X_{C,k} , \quad (3.2.13)$$

giving

$$(\epsilon_{ABC} x_{i,A} J^{-1})_{,i} = 0 , \quad (3.2.14)$$

since ϵ_{ijk} is antisymmetric in indices i, j and k and $X_{C,ik}$ and $X_{B,ij}$ are symmetric in indices i, k and i, j respectively.

Multiplying Equation 3.2.14 by ϵ_{DBC} , and using $\epsilon_{ABC} \epsilon_{DBC} = 2\delta_{AD}$, yields

$$(x_{i,A} J^{-1})_{,i} = 0 . \quad (3.2.15)$$

Thus using the second of Equation 3.2.10, the divergence of the stress tensor is written as

$$\begin{aligned}
 t_{ji,j} &= (J^{-1}x_{j,A}T_{iA})_{,j} \\
 &= J^{-1}x_{j,A}T_{iA,B}X_{B,j} \\
 &= J^{-1}\delta_{AB}T_{iA,B} \\
 &= \frac{\rho}{\rho_{00}} T_{iA,A} ,
 \end{aligned} \tag{3.2.16}$$

where it was used that $\rho J = \rho_{00}$ with ρ and ρ_{00} being the densities in the deformed (current) and initial (undeformed natural state) states respectively.

Use of Equation 3.2.16 in Equation 2.5.13, gives the form used by Truesdell [28, p. 554]

$$\rho_{00}a_i = T_{iA,A} + \rho_{00}f_i , \tag{3.2.17}$$

when the acceleration vector a_i is $\partial^2 x_i / \partial t^2$ referred to the material system.

This form is commonly used by the following authors:
Chu [15] and Davidson [18].

The stress tensor T_{iA} is in the mixed system; i.e., spatial and material coordinate system, and is not symmetric. However a completely material stress tensor T_{AB} , which is symmetric, can be introduced; it is defined by

$$T_{AB} = X_{A,i}T_{iB} = JX_{A,i}X_{B,j}t_{ij} , \tag{3.2.18}$$

with the symmetry property,

$$T_{AB} = T_{BA} , \quad (3.2.19)$$

since $t_{ji} = t_{ij}$.

To write the equation of motion in terms of T_{AB} , one may solve the first equality of Equation 3.2.18 for T_{iB} and substitute this into Equation 3.2.17. Multiplying the first equality of Equation 3.2.18 by $x_{k,A}$, one obtains

$$\begin{aligned} T_{AB} x_{k,A} &= x_{A,i} x_{k,A} T_{iB} \\ &= \delta_{ik} T_{iB} \\ &= T_{kB} . \end{aligned} \quad (3.2.20)$$

Thus Equation 3.2.17 reduces to the form

$$\rho_{00} \frac{\partial^2 x_i}{\partial t^2} = (T_{AB} x_{i,B})_{,A} + \rho_{00} f_i .$$

This is the form commonly used by the following authors: Bland [13, 14, 16, 23], Waterson [21, 22], Varley [41], Chu [15] and Davidson [18].

3.3. General Elastic Medium

The basic laws of balance for mass, Equation 2.5.12, linear momentum, Equation 2.5.13, and energy, Equation 2.5.14, hold for all continua. A particular material is further

characterized by a constitutive law. Further theories of constitutive laws characterize ideal materials. Thus when a particular theory is used to describe an actual material, the description may always be approximate. Further the same material may behave differently under different situations. Thus a theoretical description not only categorizes materials but it may also fix the situations of the same materials. It is therefore felt that any logical physical theory must stand the test of logical consistency in order that it remain scientific and the only way this can be achieved is through a physical-mathematical approach. This point is discussed because this is the philosophy under which the present study is developed.

For a general elastic medium a considerable amount of precision may be taken in defining an elastic body as one where stress is given by the instantaneous strain only [28]. One can further define an isotropic elastic medium as one for which stress is an isotropic function of strain. In earlier developments, to obtain an explicit expression a general polynomial expansion, finite or infinite (with the provision of convergence in the latter case), was assumed; then the Cayley-Hamilton theorem was used to write it in a form, which was an apparently quadratic form, with coefficients as functions of invariants of the strain tensor. However, Serrin [61] gave a proof which does not require the

assumption of a polynomial expression.

Let $\underline{e} = e_{ij}$ be a strain tensor, then a general isotropic body is defined by

$$\underline{t} = a\delta + b\underline{e} + c\underline{e}^2, \quad (3.3.1)$$

or in component form

$$t_{ij} = a\delta_{ij} + be_{ij} + ce_{ik}e_{kj}, \quad (3.3.2)$$

where a , b and c are arbitrary functions of the invariants I, II and III of e_{ij} and X_A only. The strain invariants are given by

$$I = e_{ii} = \text{Tr } \underline{e} \quad (3.3.3)$$

$$II = \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ji}) = \frac{1}{2}\{(\text{Tr } \underline{e})^2 - \text{Tr}(\underline{e}^2)\}, \quad (3.3.4)$$

$$\begin{aligned} III &= \det e_{ij} = |\underline{e}| = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}e_{ip}e_{jq}e_{kr} \\ &= \frac{1}{6}\{2e_{ij}e_{jk}e_{ki} - 3e_{ij}e_{ji}e_{kk} + e_{ii}e_{jj}e_{kk}\} \\ &= \frac{1}{6}\{2\text{Tr}(\underline{e}^3) - 3(\text{Tr } \underline{e})(\text{Tr } \underline{e}^2) + (\text{Tr } \underline{e})^3\}, \end{aligned} \quad (3.3.5)$$

with the $\text{Tr } \underline{a}$ denoting a_{ii} , the trace of the matrix a_{ij} . These expressions will be used throughout this study.

It is usual to assume, as is done here, that stress vanishes with strain. This imposes only one constraint, which is given by

$$a(0,0,0) = 0, \quad (3.3.6)$$

where $a = a(I, II, III)$. Therefore such a medium has a preferred "natural state".

3.4. Hyperelastic Medium

This is the type of medium that is the object of the present study. The complete formulation of this problem in a completely spatial configuration, as it appears at the end of the next section, is believed to be new to the literature.

A hyperelastic material is one for which stress is derivable from a potential and the stress potential is the internal energy function. More particularly the stress tensor is so defined that the energy equation, Equation 2.5.14, reduces to an identity. As a body is deformed it will acquire strain energy, the energy of deformation. It is known that for the isentropic case, strain energy and internal energy differ by only a constant; and for the isothermal case, strain energy and the Helmboltz free energy differ by the same constant [28, p. 641].

Thus to obtain the constitutive laws for a hyperelastic medium consider first the energy equation, Equation 2.5.14,

$$\rho \frac{de}{dt} = t_{ji} d_{ij} \quad (3.4.1)$$

It is conventional to write the internal energy as

$$\rho_{00} e = U \quad (3.4.2)$$

where ρ_{00} is the initial density in the undeformed (natural) state. Since e is the internal energy per unit mass, U is the internal energy per unit volume in the undeformed (natural) state.

For a general hyperelastic material, U will depend on all strains in an arbitrary manner. However the material symmetries and invariance requirements yield more and more restricted dependence. For a homogeneous, isotropic material, it depends on the strain tensor only through its invariants, I, II and III [62, p. 25] (actually, it will also depend on the entropy due to thermodynamic considerations but this will be brought out in the next section). Thus for the present time, it is assumed as

$$U = U(I, II, III). \quad (3.4.3)$$

Even after these considerations there are a multiplicity of forms of the stress-strain relations proposed. Only three of these forms, which appear to have been used in problems of wave propagation, will be noted.

Now the left member of Equation 3.4.1 can be written as

$$\frac{\rho}{\rho_{00}} \frac{dU}{dt} = \frac{\rho}{\rho_{00}} \frac{\partial U}{\partial E_{AB}} \frac{dE_{AB}}{dt} = \frac{\rho}{\rho_{00}} \frac{\partial U}{\partial E_{AB}} d_{ij} x_{i,A} x_{j,B}, \quad (3.4.4)$$

where Equation 3.1.24 has been used for dE_{AB}/dt .

Equation 3.2.18 can be rewritten as

$$t_{ij} = \frac{\rho}{\rho_{00}} x_{i,A} x_{j,B} T_{AB} \quad (3.4.5)$$

Use of Equation 3.4.5 in Equation 3.4.1 combined with Equation 3.4.4 yields

$$\frac{\rho}{\rho_{00}} \frac{\partial U}{\partial E_{AB}} d_{ij} x_{i,A} x_{j,B} = \frac{\rho}{\rho_{00}} x_{i,A} x_{j,B} T_{AB} d_{ij} . \quad (3.4.6)$$

This reduces to the definition

$$T_{AB} = \frac{\partial U}{\partial E_{AB}} , \quad (3.4.7)$$

which is attributed to Kelvin-Cosserat [59, p. 146]. This form is commonly used by Bland [23] and Waterson [21, 22].

Use of Equation 3.2.18 and the first of Equation 3.1.9 with Equation 3.4.7 gives directly another form

$$T_{iA} = \frac{\partial U}{\partial E_{AB}} x_{i,B} . \quad (3.4.8)$$

This or the one in terms of C_{AB} is commonly used by Chu [15] and Davidson [18].

Truesdell [60] in his study on principal waves uses the expression for T_{iA} in terms of $(\tilde{c}^{-1})_{ij}$. The forms used in that work are

$$T_{iA} = \frac{\rho_{00}}{\rho} t_{ij} x_{A,j} , \quad (3.4.9)$$

$$t_{ij} = a_0 \delta_{ij} + a_1 (\tilde{c}^{-1})_{ij} + a_2 (\tilde{c}^{-1})_{ik} (\tilde{c}^{-1})_{kj} , \quad (3.4.10)$$

where a_0 , a_1 and a_2 are functions of the invariants of $(c^{-1})_{ij}$, which is given in Equation 3.1.5. The concept of principal waves will be discussed later and so the above forms are given here for later reference.

Consider now the completely spatial form. Use of Equation 3.1.25, i.e.,

$$\frac{de_{kl}}{dt} = d_{kl} - (e_{km}v_{m,l} + e_{lm}v_{m,k}), \quad (3.4.11)$$

enables one to write Equation 3.4.1 as

$$\begin{aligned} t_{ji}v_{i,j} &= \frac{\rho}{\rho_{00}} \frac{\partial U}{\partial e_{kl}} \frac{de_{kl}}{dt} \\ &= \frac{\rho}{\rho_{00}} \frac{\partial U}{\partial e_{kl}} \{v_{kl} - (e_{km}v_{m,l} + e_{lm}v_{m,k})\} \\ &= \frac{\rho}{\rho_{00}} \frac{\partial U}{\partial e_{kl}} \{\delta_{ik}\delta_{lj} - e_{km}\delta_{mi}\delta_{lj} - e_{lm}\delta_{mi}\delta_{kj}\}v_{i,j} . \end{aligned} \quad (3.4.12)$$

Thus the stress tensor can be defined as

$$\begin{aligned} t_{ij} &= \frac{\rho}{\rho_{00}} \left\{ \frac{\partial U}{\partial e_{kj}} \delta_{ik} - 2e_{ik} \frac{\partial U}{\partial e_{kj}} \right\} \\ &= \frac{\rho}{\rho_{00}} (\delta_{ik} - 2e_{ik}) \frac{\partial U}{\partial e_{kj}} . \end{aligned} \quad (3.4.13)$$

Here symmetry of e_{ij} and d_{ij} is used at various stages.

This is the only form of the constitutive law for an isotropic elastic material in completely spatial form with e_{ij} given by Equation 3.1.7 and will be used throughout this study.

An important feature of such a definition of stress; i.e., that it is derivable from a potential such that it reduces the energy equation to an identity, is that one can add any tensor, whose inner product with d_{ij} is zero and thus contributes nothing to the energy equation. So the stress tensor retains only that part which contributes to change in the strain energy.

Some relationships for isotropic, homogeneous materials will be collected here for later use (it is to be noted that the assumption of homogeneity is made here for analytical convenience; it is later indicated that the analysis presented here is also valid if the material is nonhomogeneous). Firstly the material functions a , b and c in Equation 3.3.2 will be evaluated. To this end, using the definitions of the strain invariants in Equations 3.3.3, 3.3.4 and 3.3.5, the following identities are easily verified

$$\frac{\partial I}{\partial e_{kj}} = \delta_{kj} ; \quad \frac{\partial II}{\partial e_{kj}} = I\delta_{kj} - e_{kj} , \quad (3.4.14)$$

$$\frac{\partial III}{\partial e_{kj}} = e_{k\ell}e_{\ell j} - Ie_{kj} + II\delta_{kj} . \quad (3.4.15)$$

Since, for an isotropic, homogeneous material, the internal energy function U depends only on the invariants I , II and III , one has

$$\frac{\partial U}{\partial e_{kj}} = U_1 \delta_{kj} + U_2 (I \delta_{kj} - e_{kj}) + U_3 (e_{kl} e_{lj} - I e_{kj} + II \delta_{kj}), \quad (3.4.16)$$

with suffixes on U denoting partial derivatives with respect to the invariants; e.g.,

$$U_1 = \left. \frac{\partial U}{\partial I} \right|_{II, III = \text{constant}} \quad (3.4.17)$$

Use of Equation 3.4.16 in Equation 3.4.10 and simplifying, yields

$$\begin{aligned} \frac{\rho_{00}}{\rho} t_{ij} = & U_1 \delta_{ij} + U_2 (I \delta_{ij} - e_{ij}) + U_3 (e_{il} e_{lj} - I e_{ij} + II \delta_{ij}) \\ & - 2\{U_1 e_{ij} + U_2 (I e_{ij} - e_{ik} e_{kj}) \\ & + U_3 (e_{ik} e_{kl} e_{lj} - I e_{ik} e_{kj} + II e_{ij})\}. \end{aligned} \quad (3.4.18)$$

Use of the Cayley-Hamilton theorem, i.e.,

$$e_{ik} e_{kl} e_{lj} - I e_{ik} e_{kj} + II e_{ij} - III \delta_{ij} = 0, \quad (3.4.19)$$

in Equation 3.4.18, yields after simplifying

$$\begin{aligned} \frac{\rho_{00}}{\rho} t_{ij} = & \delta_{ij} \{U_1 + I U_2 + (II - 3III) U_3\} \\ & + e_{ij} \{-2U_1 - (2I + 1) U_2 - I U_3\} + e_{ik} e_{kj} \{U_3 + 2U_2\}. \end{aligned} \quad (3.4.20)$$

Thus comparing with Equation 3.3.2, one has

$$a = \frac{\rho}{\rho_{00}} \{U_1 + I U_2 + (II - 3III) U_3\};$$

$$b = \frac{\rho}{\rho_{oo}} \{-2U_1 - (2I+1)U_2 - IU_3\};$$

$$c = \frac{\rho}{\rho_{oo}} \{2U_2 + U_3\}. \quad (3.4.21)$$

Another form of Equation 3.4.20 which will be used extensively in later sections of this study is

$$\begin{aligned} \frac{\rho_{oo}}{\rho} t_{ij} = & U_1 \{\delta_{ij} - 2e_{ij}\} + U_2 \{I\delta_{ij} - (2I+1)e_{ij} + 2e_{ik}e_{kj}\} \\ & + U_3 \{(II-2III)\delta_{ij} - Ie_{ij} + e_{ik}e_{kj}\}. \end{aligned} \quad (3.4.22)$$

Lastly, it is to be noted that density is now not an independent variable; it can be expressed in terms of the strain invariants as

$$\rho = \rho_{oo} \{1 - 2I + 4II - 8III\}^{1/2}. \quad (3.4.23)$$

Equation 3.3.2 with a , b and c given by Equations 3.4.21 and ρ given by Equation 3.4.23 completes the statement of the constitutive law for an isotropic, homogeneous, hyper-elastic material.

3.5. Thermodynamic Considerations

By requiring that the energy equation reduce to an identity for an appropriate definition of the stress tensor, the above considerations gave the constitutive law. But the internal energy function (strain energy) does not act as a

potential in a simple way to derive the stress components in a purely spatial formulation, as it does, for example, in the case of the purely material formulation given in the previous section. In order to study admissibility conditions for shocks by the requirement that the entropy must increase across the shock wave, the stress tensor must appear as the partial derivative of the internal energy function with respect to strain with entropy held constant. To bring out this concept of stress in the thermodynamic sense, let us first illustrate by what is done in gas-dynamics.

The shock conditions for gas-dynamics are obtained from Equations 2.5.16, 2.5.17 and 2.5.18 by defining $t_{ij} = -P\delta_{ij}$, one then obtains

$$\rho(G - v_n) = \rho_0(G - v_{on}) = m_0, \quad (3.5.1)$$

$$m_0[v_i] = [P]n_i \quad (3.5.2)$$

$$m_0\left[e + \frac{1}{2}v^2\right] = [Pv_n], \quad (3.5.3)$$

where ρ, ρ_0 are densities; P is the pressure; v_i is the velocity vector, n_i is the unit normal vector with $v_n = v_i n_i$ denoting the normal component of v_i ; $v^2 = v_i v_i$ is the square of the magnitude of the velocity; G is the shock normal speed of propagation and e is the internal energy per unit mass.

The most basic difference between elastic shocks and

gas-dynamical shocks is that, in gas-dynamics, the field variables themselves are discontinuous so that the definition of a unique normal shock speed G and a unique normal vector n_i assigns a unique meaning to all quantities used. However in the case of elasticity, it is the derivatives that are discontinuous and because of this the shock conditions will involve not only G and n_i as above but also, possibly, curvatures. Thus, if one uses a material description of the shock surface, besides the disadvantage in physical interpretation in such a description, the terms themselves, viz. curvatures, must be assigned a unique meaning. When the basic space-metric itself is subjected to discontinuities, which is the case with a material description of a shock, and the fields are defined in terms of the underlying metric, then such a description appears quite unrealistic.

Continuing with the example of gas-dynamics, one can obtain from Equations 3.5.1, 3.5.2 and 3.5.3 the well known Hugoniot relation

$$e - e_0 + \frac{(P + P_0)}{2} (\tau - \tau_0) = 0; \quad \tau = \frac{1}{\rho}. \quad (3.5.4)$$

The most important feature of this relation is that it does not contain the velocity vector or G ; thus it is a relation between thermodynamic quantities only. Further it is a relation involving only thermodynamic state functions.

One supplements Equation 3.5.4 with Gibb's Equation

$$de = \theta dS - P d\tau, \quad (3.5.5)$$

where θ is the absolute temperature and S is the entropy per unit mass.

The basic assumption of thermodynamics is that any three thermodynamic state variables are related. Therefore any three of the thermodynamic variables e , θ , P , τ and S are related. Combining this with Equations 3.5.4 and 3.5.5 allows one to regard all the functions involved as functions of a single variable taken in gas-dynamics to be τ . Then combining Equations 3.5.4 and 3.5.5 one can assert

$$S - S_0 = - \frac{1}{12\theta_0} \frac{\partial^2 P}{\partial \tau^2} \bigg|_0 (\tau - \tau_0)^3 + O(\tau - \tau_0)^4 + \dots \quad (3.5.6)$$

Further one imposes the physical requirement that entropy must increase across the shock (this strict inequality of entropy increase is proved from the condition that shocks are limit solutions of a viscous, heat-conducting gaseous medium [1]) which implies, from Equation 3.5.6 that, for materials with $P_{\tau\tau}|_0 > 0$, shocks must be compressive ($\tau_0 > \tau$) and that entropy changes are third order in the strength of the shock, as measured here by $\tau - \tau_0$.

The above assertions are true only for weak shock waves since it is assumed that further terms in the Taylor series in Equation 3.5.6 may be neglected. It is noted

that a more general result is available in gas-dynamics. The first aim here is to obtain a result analogous to Equation 3.5.6 in elasticity and to this end a relation analogous to Equation 3.5.5 is needed.

To introduce these concepts, one first gathers from thermodynamics the considerations that: A total of $n+1$ independent parameters S, v_α ($\alpha=1,2,\dots,n$) determine the internal energy e as [59, p. 119], [28, p. 619]

$$e = e(S, v_\alpha), \quad (3.5.7)$$

where S is the entropy per unit mass and v_α are called substate parameters.

From Equation 3.5.7, one defines temperature and the thermodynamic tensions f_α as

$$\theta = \left. \frac{\partial e}{\partial S} \right|_{v_\alpha = \text{constant}}; \quad f_\alpha = \left. \frac{\partial e}{\partial v_\alpha} \right|_{S, v_\beta (\beta \neq \alpha) = \text{constant}}. \quad (3.5.8)$$

This leads to Gibb's equation

$$de = \theta ds + f_\alpha dv_\alpha. \quad (3.5.9)$$

In gas-dynamics there is only one thermodynamic tension f_α ($f_1 = -P$) and one substate parameter v_α ($v_1 = \tau$). By analogy in elasticity one considers the stress tensor as giving six tensions and the components of the strain tensor providing six substate parameters.

Two immediate forms of this type follow from Equation

3.4.8, first note the result

$$\frac{\partial}{\partial x_{i,A}} = x_{i,B} \frac{\partial}{\partial E_{AB}}, \quad (3.5.10)$$

since $2E_{AB} = x_{i,A} x_{i,B}^{-\delta_{AB}}$.

Then one can write Equations 3.4.8 and 3.4.7 as

$$T_{iA} = \left. \frac{\partial U}{\partial x_{i,A}} \right|_{S=\text{constant}}; \quad \rho_{00} e = U, \quad (3.5.11)$$

$$= \rho_{00} \left. \frac{\partial e}{\partial x_{i,A}} \right|_S, \quad (3.5.12)$$

and

$$T_{AB} = \rho_{00} \left. \frac{\partial e}{\partial E_{AB}} \right|_S. \quad (3.5.13)$$

Thus using T_{iA} (Piola-Kirchoff) and T_{AB} (Kelvin-Cosserat) as the stress tensors one can have definitions of stress tensors which reduce the energy equation and Gibb's equations to identities. These however are mixed spatial-material and pure material forms respectively. The aim here is to obtain a pure spatial form.

Now it is difficult to see how to write Equation 3.4.13 as a derivative of U with respect to e_{ij} , holding S constant, in a form similar to Equations 3.5.12 and 3.5.13. The derivative of U with respect to e_{ij} holding S constant is believed to be a completely new result of this study.

To obtain this result consider first another strain tensor, which is attributed to Ludwik [63] and has been used by Hencky [64]. It is known as the Hencky strain measure or

natural strain and is defined as¹⁰

$$h = -\frac{1}{2} \log(1-2e); \quad 2e = 1 - \exp(-2h), \quad (3.5.14)$$

$$2h = \sum_{n=1}^{\infty} \frac{(2e)^n}{n}; \quad 2e = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2h)^n}{n!}, \quad (3.5.15)$$

and convergence of Equation 3.5.15 requires $\max|2e_{ij}| < 1$.

Here e and h represent e_{ij} and h_{ij} respectively. Though one may object to questions of convergence with regard to Equations 3.5.14 (see however [28, p. 269], Equations 3.5.15 are free from it; in this study, what is implied by the logarimetric and exponential symbols is the series itself.

It is well known that if e is diagonal, so is e^n . Thus referred to the principal axes of e , all the powers of e have only diagonal components. Thus for principal directions the series reduces to

$$2h_a = \sum_{n=1}^{\infty} \frac{(2e_a)^n}{n}; \quad 2e_a = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2h)^n}{n!}, \quad (3.5.16)$$

where the suffixes a,b,c are used to denote principal values only and range over 1, 2 and 3.

Thus the principal directions of h and e coincide and for principal values the closed form of sums is given by

$$2h_a = -\log(1-2e_a); \quad 2e_a = 1 - \exp(-2h_a). \quad (3.5.17)$$

¹⁰The sumbol Σ is used both for denoting summation and for the singular surface; however context is enough to indicate the meanings.

An isotropic elastic material is one for which the principal axes of the stress and strain tensor coincide. So referred to principal axes of \underline{e} , the stress tensor is diagonal with principal values t_a .

Then referred to principal axes, Equation 3.1.25 can be written as

$$\frac{de_1}{dt} = (1-2e_1)d_{11} ; \frac{de_2}{dt} = (1-2e_2)d_{22}, \frac{de_3}{dt} = (1-2e_3)d_{33}, \quad (3.5.18)$$

$$0 = d_{12} - (e_1 v_{1,2} + e_2 v_{2,1}); \quad 0 = d_{23} - (e_2 v_{2,3} + e_3 v_{3,2}),$$

$$0 = d_{31} - (e_3 v_{3,1} + e_1 v_{1,3}) . \quad (3.5.19)$$

This is the most crucial result. The principal axes of the material derivative of \underline{e} and the deformation tensor \underline{d} (alone) do not coincide. Thus \underline{d} is not diagonal when \underline{e} is diagonal.

The system of Equations 3.5.18 gives

$$\frac{dh_a}{dt} = d_{aa} = v_{a,a} , \quad (a=1,2,3; \text{ no summation on } a's). \quad (3.5.20)$$

However it does not lead to the relation

$$\frac{dh_{ij}}{dt} = d_{ij} . \quad (3.5.21)$$

To assert this, consider Equation 3.1.25 in the following forms

$$\begin{aligned}\frac{de_{ij}}{dt} &= \dot{d}_{ij} - (e_{im}\dot{d}_{mj} + \dot{d}_{im}e_{mj}) - (e_{im}\dot{\Omega}_{mj} - \dot{\Omega}_{im}e_{mj}) \\ &= \dot{d} - (\dot{e}d + d\dot{e}) - (\dot{e}\Omega - \Omega\dot{e}),\end{aligned}\quad (3.5.22)$$

where

$$v_{i,j} = \dot{d}_{ij} + \Omega_{ij}; \quad 2\Omega_{ij} = v_{i,j} - v_{j,i}. \quad (3.5.23)$$

One can now consider the first few powers of Equation 3.5.22

$$\frac{de^2}{dt} = (\dot{e}d + d\dot{e}) - (e^2\dot{d} + 2e\dot{e}d + d\dot{e}^2) - (e^2\dot{\Omega} - \Omega\dot{e}^2);$$

$$\begin{aligned}\frac{de^3}{dt} &= (e^2\dot{d} + e\dot{e}d + d\dot{e}^2) - (e^3\dot{d} + 2e^2\dot{e}d + 2e\dot{e}d^2 + d\dot{e}^3) \\ &\quad - (e^3\dot{\Omega} - \Omega\dot{e}^3); \end{aligned}$$

$$\begin{aligned}\frac{de^4}{dt} &= (e^3\dot{d} + e^2\dot{e}d + e\dot{e}d^2 + d\dot{e}^3) - (e^4\dot{\Omega} - \Omega\dot{e}^4) \\ &\quad - (e^4\dot{d} + 2e^3\dot{e}d + 2e^2\dot{e}d^2 + 2e\dot{e}d^3 + d\dot{e}^4).\end{aligned}\quad (3.5.24)$$

Thus the succeeding terms, though showing a pattern, do not cancel out. Therefore the assertion that Equation 3.5.21 is not valid is shown. However when $e_{ij} (\equiv \underline{e})$ is

diagonal, the last bracketed terms of Equations 3.5.24 do not contribute since \underline{e} has only diagonal components and \underline{Q} has these components vanishing. Moreover for \underline{e} diagonal Equations 3.5.24 yield the following forms

$$\begin{aligned}\frac{de_1}{dt} &= d_{11} - 2e_1 d_{11} = (1 - 2e_1) d_{11} ; \\ \frac{de_1^2}{dt} &= 2e_1 (1 - 2e_1) d_{11} ; \\ \frac{de_1^3}{dt} &= 3e_1^2 (1 - 2e_1) d_{11} ; \\ \frac{de_1^n}{dt} &= ne_1^{n-1} (1 - 2e_1) d_{11} ,\end{aligned}\tag{3.5.25}$$

with similar results for e_2 and e_3 using Equations 3.5.18.

Thus the failure of Equation 3.5.21 is not only due to the existence of rotations \underline{Q} but is more involved.

This failure of Equation 3.5.21 forces one to formulate the constitutive laws, when using a completely spatial description, in terms of principal values. The energy equation, Equation 3.4.1, can now be written as

$$\frac{\rho}{\rho_{00}} \frac{dU}{dt} = t_1 \frac{dh_1}{dt} + t_2 \frac{dh_2}{dt} + t_3 \frac{dh_3}{dt} .\tag{3.5.26}$$

Thus the constitutive laws in a purely spatial description are given by

$$t_a = \frac{\rho}{\rho_{oo}} \frac{\partial U}{\partial h_a}, \quad (3.5.27)$$

or

$$t_a = \frac{\rho}{\rho_{oo}} (1-2e_a) \frac{\partial U}{\partial e_a}, \quad (3.5.28)$$

which is the form of Equation 3.4.13 referred to principal directions and now $U=U(I,II,III,S)$.

Now the Gibb's equation, Equation 3.5.9, can be written as

$$\frac{1}{\rho_{oo}} dU = \theta ds + \frac{1}{\rho} \{t_1 dh_1 + t_2 dh_2 + t_3 dh_3\}, \quad (3.5.29)$$

or

$$dU = \rho_{oo} \theta ds + \frac{\rho_{oo}}{\rho} \left\{ \frac{t_1 de_1}{1-2e_1} + \frac{t_2 de_2}{1-2e_2} + \frac{t_3 de_3}{1-2e_3} \right\}. \quad (3.5.30)$$

This asserts that the constitutive laws, Equation 3.5.27 are derivatives of the internal energy with respect to strain with entropy held constant.

In the deformed configuration the constitutive law, given by Equation 3.5.28 is stated for principal directions only. Further one can write Equation 3.5.28 in the following form

$$\begin{aligned} t_1 &= a + be_1 + ce_1^2; & t_2 &= a + be_2 + ce_2^2, \\ t_3 &= a + be_3 + ce_3^2, \end{aligned} \quad (3.5.31)$$

where a , b and c have the same values as in Equation 3.4.21.

From Equation 3.4.13, the constitutive law in the deformed configuration can also be written as

$$t_{ij} = a\delta_{ij} + be_{ij} + ce_{ik}e_{kj} . \quad (3.5.32)$$

It is asserted that Equations 3.5.31 and 3.5.32 are equivalent.

Given the strain tensor (e_{ij}) , the principal values and directions are given by

$$e_{ij}l_j^a = e_a l_i^a; \quad l_i^a l_i^b = \delta_{ab}; \quad (a \text{ not summed}), \quad (3.5.33)$$

where a, b range over 1, 2 and 3.

Multiply Equation 3.5.32 by $l_i^1 l_j^1$ to obtain

$$\begin{aligned} t_{ij} l_i^1 l_j^1 &= a l_i^1 l_j^1 + b l_i^1 (e_{ij} l_j^1) + c (e_{ik} l_i^1) (e_{kj} l_j^1) \\ &= a l_i^1 l_j^1 + b l_i^1 (e_1 l_i^1) + c (e_1 l_k^1) (e_1 l_k^1), \end{aligned} \quad (3.5.34)$$

which is the same as t_1 from the definition of the stress tensor in terms of its principal values and directions; which are given by

$$t_{ij} l_j^a = t_a l_i^a; \quad l_i^a l_i^b = \delta_{ab}; \quad (a \text{ not summed}), \quad (3.5.35)$$

where a, b range over 1, 2 and 3 again.

One of course obtains similar results for t_2 and t_3 .

Conversely one can start with

$$\begin{aligned} t_{ij} &= t_1 l_i^1 l_j^1 + t_2 l_i^2 l_j^2 + t_3 l_i^3 l_j^3 \\ &= (a + b e_1 + c e_1^2) l_i^1 l_j^1 + (a + b e_2 + c e_2^2) l_i^2 l_j^2 \end{aligned}$$

$$\begin{aligned}
& + (a + be_3 + ce_3^2) l_i^3 l_j^3 \\
& = a \{ l_i^1 l_j^1 + l_i^2 l_j^2 + l_i^3 l_j^3 \} \\
& \quad + b \{ e_1 l_i^1 l_j^1 + e_2 l_i^2 l_j^2 + e_3 l_i^3 l_j^3 \} \\
& \quad + c \{ e_1^2 l_i^1 l_j^1 + e_2^2 l_i^2 l_j^2 + e_3^2 l_i^3 l_j^3 \} \\
& = a \{ \sum_a l_i^a l_j^a \} + b e_{ik} \{ \sum_a l_k^a l_j^a \} + c e_{ik} e_{jl} \{ \sum_a l_k^a l_l^a \} .
\end{aligned} \tag{3.5.36}$$

Now (l_i^1, l_i^2, l_i^3) are unit vectors in the three orthogonal directions referred to the x_i coordinate system. So one has, for example;

$$l_i^1 l_i^1 = 1; \quad l_i^1 l_i^2 = 0; \text{ etc.} \tag{3.5.37}$$

One further has

$$\sum_a l_i^a l_j^a = \delta_{ij}, \quad (a=1,2,3) . \tag{3.5.38}$$

This is just a statement that referred to principal axes, the direction cosines of x_i system form an orthonormal triad.

Thus the equivalence of Equations 3.5.31 and 3.5.32 is proven; this is equivalent to saying that Equations 3.4.13 and 3.5.28 lead to the same constitutive law.

Entropy changes across shocks in an incompressible, isotropic medium is also discussed later in this study. In order to study shocks in an incompressible medium, one has to

first modify the constitutive law. Returning to Gibb's equation, Equation 3.5.30, and the energy equation, Equation 3.4.1,

$$dU = \rho_{oo} \theta dS + \frac{t_1 de_1}{1-2e_1} + \frac{t_2 de_2}{1-2e_2} + \frac{t_3 de_3}{1-2e_3} ;$$

$$\frac{\rho}{\rho_{oo}} \frac{dU}{dt} = t_{ij} d_{ij} = t_1 d_{11} + t_2 d_{22} + t_3 d_{33}.$$

However one also has

$$\begin{aligned} \frac{\rho}{\rho_{oo}} &= \{1-2I+4II-8III\}^{1/2} \\ &= \{(1-2e_1)(1-2e_2)(1-2e_3)\}^{1/2} . \end{aligned} \quad (3.5.39)$$

Thus incompressibility imposes the following constraint

$$\rho = \rho_{oo}; \quad I-2II+4III = 0;$$

$$d_{ii} = \frac{dh_i}{dt} = 0; \quad \frac{de_1}{1-2e_1} + \frac{de_2}{1-2e_2} + \frac{de_3}{1-2e_3} = 0, \quad (3.5.40)$$

and in terms of h_a one has

$$\begin{aligned} \log \frac{\rho}{\rho_{oo}} &= \frac{1}{2} \{ \log(1-2e_1) + \log(1-2e_2) + \log(1-2e_3) \} \\ &= -(h_1+h_2+h_3) = 0, \end{aligned} \quad (3.5.41)$$

which gives the result

$$I_h = h_1+h_2+h_3 = 0 . \quad (3.5.42)$$

Multiplying the last two forms of the constraint equation, Equation 3.5.40, by a scalar P and adding these to Equations 3.4.1 and 3.5.29 respectively, one has

$$\begin{aligned}\frac{dU}{dt} &= (t_1+P) \frac{dh_1}{dt} + (t_2+P) \frac{dh_2}{dt} + (t_3+P) \frac{dh_3}{dt} \\ &= (t_1+P)d_{11} + (t_2+P)d_{22} + (t_3+P)d_{33},\end{aligned}\tag{3.5.43}$$

and

$$\begin{aligned}dU &= \rho_{00}\theta dS + (t_1+P)dh_1 + (t_2+P)dh_2 + (t_3+P)dh_3 \\ &= \rho_{00}\theta dS + \frac{(t_1+P)de_1}{1-2e_1} + \frac{(t_2+P)de_2}{1-2e_2} + \frac{(t_3+P)de_3}{1-2e_3}.\end{aligned}\tag{3.5.44}$$

The indeterminate scalar multiplier P does no work and is called pressure. Now the dependence of U on the invariants of e_{ij} can be taken to be only through any two of I, II and III. It is only in the case of the invariants of h_a that I_h is zero, and thus omitted. This lack of uniqueness in dependence of U on I, II and III is trivial; dependence on any two of them is general enough, since they are always related by the second of Equation 3.5.40. For definiteness U is taken now to depend on the first two invariants of e_{ij} . The constitutive law is then

$$t_a = -P + \frac{\partial U}{\partial h_a}$$

$$= -P + (1-2e_a) \frac{\partial U}{\partial e_a}, \quad (a \text{ not summed}), \quad (3.5.45)$$

where $U=U(I,II,S)$ and the suffix (a) ranges over 1, 2 and 3.

The general tensorial form of the constitutive law for an incompressible material satisfying Equation 3.4.1 is now clearly given by

$$t_{ij} = -P\delta_{ij} + (\delta_{ik}-2e_{ik}) \frac{\partial U}{\partial e_{kj}}, \quad (3.5.46)$$

$$= (-P+a)\delta_{ij} + be_{ij} + ce_{ik}e_{kj}. \quad (3.5.47)$$

Refer Equation 3.5.47 to principal directions and comparing with the second of Equation 3.5.45, the material functions b and c, given in Equation 3.4.21, become

$$b = -2U_1 - (2I+1)U_2; \quad c = 2U_2. \quad (3.5.48)$$

It is to be stressed at this stage that for an incompressible, hyperelastic and isotropic material, since $U=U(I,II,S)$, there are only two material functions, U_1 and U_2 or b and c, which are given in Equation 3.5.48. Further the material function a in the elastic constitutive equation is of no significance here; it is absorbed in the pressure P; since P is indeterminate, $P-a$ can be considered also as the new indeterminate pressure.

The introduction of such principal values complicates the shock problem quite a lot; but this seems to be the only

unique formulation in the deformed (spatial) system. It has to be recognized that the complication is only an algebraic one. It has a minimal character, the tensions (t_a/ρ) and substates (h_a) are now three in number instead of six in the other formulations.

The appearance of h_a as a strain measure is forced on the formulation from thermodynamical considerations. One can see in the current literature a search for an appropriate strain measure so that the elastic stress-strain law is linear over a larger range of strains. The measure h_a stands as a good contestant in this competition [65]. That it should arise in a thermodynamical formulation too is quite a coincidence.

For the isotropic case use of principal values presents no difficulties as explained subsequently. However, in the anisotropic case when the principal axes of stress and strain no longer coincide, it does not seem possible to find another strain measure which leads to a precise constitutive law for the stress tensor so as to reduce both the energy equation and the Gibb's equation to identities. So another way is to seek another pseudo-stress which is of completely spatial character. To this end one defines a tensor, b_{ij} , as the inverse of $a_{ij} = \delta_{ij} - 2e_{ij}$; this is given by

$$\Delta b_{ij} = (1 - 2I + 4III)\delta_{ij} + 2(1 - 2I)e_{ij} + 4e_{ik}e_{kj}, \quad (3.5.49)$$

$$\Delta = 1-2I+4II-8III; \underline{a}\underline{b}=\underline{b}\underline{a}=\underline{\delta}. \quad (3.5.50)$$

Then rewrite Gibb's equation with the pseudo-stress tensor, $\frac{1}{\rho} b_{ik} t_{kj}$, as the thermodynamic tensions in the form

$$\frac{1}{\rho_{oo}} dU = \theta dS + \frac{1}{\rho} b_{ik} t_{kj} de_{ij}, \quad (3.5.51)$$

which leads to the definition of the thermodynamic tension corresponding to the substate parameters, e_{ij} , as

$$b_{ik} t_{kj} = \frac{\rho}{\rho_{oo}} \frac{\partial U}{\partial e_{ij}}. \quad (3.5.52)$$

Premultiplying Equation 3.5.52 by a_{mi} , one obtains

$$t_{mj} = \frac{\rho}{\rho_{oo}} a_{mi} \frac{\partial U}{\partial e_{ij}}, \quad (3.5.53)$$

which is the last of Equation 3.4.13.

Thus Equations 3.5.28 and 3.5.52 give the complete spatial formulation needed in this study. These results are believed to be new. While Equation 3.5.52 involves a more complicated expression, it is valid for anisotropic cases too; however the simple form given in Equation 3.5.28 is valid only for isotropic materials.

It is to be noted that the dependence of U on the strain invariants for the general case of anisotropy depends on the type of anisotropy [62]. Thus the dependence of U on e_{ij} is only through the invariants of e_{ij} which are appropriate for the type of anisotropy considered.

The formulation has been conceptually clearer. It is the only formulation in the deformed (spatial) system as a reference system. Further with the spatial system as basic, it is free of the ambiguities that persist in a general formulation based on the material system. However, the formulation in the spatial system may seem lengthy but this complication is one of purely algebraic character; it has, on the contrary, conceptual clarity, devoid of ambiguity and is physically meaningful. No doubt more work is needed to achieve as much understanding of elastic shocks as that of gas-dynamical shocks.

4. ENTROPY INEQUALITIES: ISOTROPIC MEDIUM-INITIALLY UNSTRAINED AND AT REST

4.1. General Considerations

As discussed in Chapter 2 of this study, shock conditions do not lead to unique solution. One has to further impose conditions of admissibility. Amongst these, the one involving the basic thermodynamic principle of increase of entropy is quite an important one. This is the one that is imposed here. One would like to assert such conditions for shocks of arbitrary strength, as is done in gas-dynamics; it has not been possible to do this here in such generality. Such a study needs more detailed knowledge of the internal energy function. However, the inequalities obtained are valid for weak but finite amplitude shock waves in a hyper-elastic medium and it sheds some light on what are commonly called higher-order elasticities.

4.2. Shock Conditions and the Generalized Rankine-Hugoniot Relation

The generalized Rankine-Hugoniot relation is a relation between purely thermodynamic quantities. This relation is first obtained for the general case of a arbitrary curved three-dimensional shock wave moving into an arbitrarily, initially strained compressible medium in motion. It is obtained from the shock conditions alone and thus the only

assumptions are those used in the derivation of the shock conditions. All earlier studies [17, 19, 23] obtain it only for plane shocks in an initially unstrained medium at rest.

To derive this relation, one first needs the shock conditions given in Equations 2.5.16, 2.5.17, and 2.5.18; these are:

$$\rho(v_n - G) = \rho_o(v_{on} - G), \quad (4.2.1)$$

$$\rho_o(v_{on} - G)[v_i] = [t_{ji}]n_i, \quad (4.2.2)$$

$$\rho_o(v_{on} - G)\left\{[e] + \frac{1}{2}[v_i v_i]\right\} = [t_{ji}v_i]n_j, \quad (4.2.3)$$

$$[v_n] = [v_i]n_i, \quad (4.2.4)$$

where the suffix 1 on the field variables in the shocked medium is omitted above and throughout and the suffix o indicates the value of the field variables in the unshocked medium.

Firstly one can obtain another form of Equation 4.2.1. Add $-\rho_o(v_n - G)$ to Equation 4.2.1 to obtain

$$[v_n] = - \frac{(v_{on} - G)[\rho]}{\rho}. \quad (4.2.5)$$

Multiply Equation 4.2.2 by n_i and make use of Equation 4.2.5 to obtain

$$[t_{nn}] = [t_{ji}]n_i n_j = - \frac{\rho_o(v_{on} - G)^2[\rho]}{\rho}. \quad (4.2.6)$$

Now using the bracketed operation given in Equation

2.5.22 with Equations 4.2.2 and 4.2.3 leads to

$$\rho_o(v_{on}-G)[e] + \rho_o(v_{on}-G)\bar{v}_i[v_i] = \bar{t}_{ji}[v_i]n_j + \bar{v}_i[v_i]\rho_o(v_{on}-G),$$

or

$$\rho_o(v_{on}-G)[e] = \bar{t}_{ji}[v_i]n_j. \quad (4.2.7)$$

Lastly, use of Equations 4.2.5 and 4.2.6 in Equation 4.2.7 leads to the final form

$$[e] = - \frac{\bar{t}_{ji}[t_{ki}]n_j n_k [\rho]}{\rho_o \rho [t_{nn}]}$$

or

$$[U] = - \frac{\rho_{oo}[\rho](t_{ji}+t_{oji})[t_{ki}]n_j n_k}{2\rho_o \rho [t_{nn}]} . \quad (4.2.8)$$

This is then the generalized Rankine-Hugoniot relation valid for any compressible continuum. As indicated previously its only assumptions are those assumed in the derivations of the shock conditions and $[\rho] \neq 0$.

It is easily verified that Equation 4.2.8 reduces to the Hugoniot relation in gas-dynamics. For gas-dynamics $t_{ij} = -P\delta_{ij}$, where P is the pressure; substituting this into Equation 4.2.8 leads to

$$[e] = \frac{[P](P+P_o)}{2\rho_o \rho} = \frac{(P+P_o)(\tau_o - \tau)}{2}, \quad (4.2.9)$$

where $\tau = 1/\rho$, which is the Hugoniot relation given in Equation 3.5.4.

From Equation 4.2.8, more restrictive forms of the generalized Rankine-Hugoniot relation can be obtained. One such form will be obtained here for later use, which is valid for an arbitrary curved 3-dimensional shock moving in an arbitrary but uniformly strained medium. From this reduced form of Equation 4.2.8 all other forms needed in this study can be obtained. Further, most of the general relations needed in this study will be obtained here.

Since the initial strain is uniform one can choose the reference spatial rectangular coordinate axes to coincide with the principal axes of the initial strain. Here one has to distinguish three densities: the density in the undeformed natural state is denoted by ρ_{00} as before; the density in the unshocked but initially strained material ahead of the shock wave is denoted by ρ_0 ; lastly the density of the shocked material behind the shock wave is denoted by ρ .

Referred to this choice of axes, the strain components in the unshocked material can be taken as being given in terms of the displacement gradients, as

$$\begin{aligned}
 u_{oi,j} &\neq 0, i=j; \quad u_{oi,j} = 0, i \neq j, \\
 e_{o11} &= u_{o1,1} - \frac{1}{2}u_{o1,1}^2; \quad e_{o22} = u_{o2,2} - \frac{1}{2}u_{o2,2}^2, \\
 e_{o33} &= u_{o3,3} - \frac{1}{2}u_{o3,3}^2; \quad e_{oij} = 0, i \neq j.
 \end{aligned} \tag{4.2.10}$$

The compatibility conditions, given in Equations 2.3.14 and 2.3.16, are for $[Z]=0$

$$[Z] = 0; \quad [Z_{,i}]n_i = B, \quad (4.2.11)$$

and

$$[Z_{,i}] = Bn_i; \quad \left[\frac{\partial Z}{\partial t}\right] = -GB. \quad (4.2.12)$$

Then replacing Z in Equations 4.2.11 and 4.2.12 by the displacements u_i , one obtains the compatibility conditions for discontinuities in the displacement gradients as

$$[u_i] = 0; \quad [u_{i,j}] = \xi_i n_j; \quad \left[\frac{\partial u_i}{\partial t}\right] = -G\xi_i. \quad (4.2.13)$$

The first secures the continuity of the material, the second gives the discontinuity of strain, and the third gives the discontinuity of the local time rate of change of displacement. First one needs the discontinuities in the velocity components. Note the definition of the velocity vector, with u_i as basic spatial variables, as

$$v_i = \frac{\partial u_i}{\partial t} + u_{i,j}v_j. \quad (4.2.14)$$

Taking the jump in the velocity vector and making use of the bracketed operation defined in Equation 2.5.20, one obtains

$$\begin{aligned} [v_i] &= \left[\frac{\partial u_i}{\partial t}\right] + [v_j u_{i,j}] \\ &= \left[\frac{\partial u_i}{\partial t}\right] + [v_j][u_{i,j}] + v_{oj}[u_{i,j}] + u_{oi,j}[v_j]. \end{aligned} \quad (4.2.15)$$

Use of Equation 4.2.13 in Equation 4.2.15 and $[v_i]=v_i$, since $v_{oi}=0$, yields

$$v_i = -G\xi_i + v_i(\xi_i n_j + u_{oi,j}) . \quad (4.2.16)$$

For simplicity of operation, one defines the following reduced discontinuities

$$\bar{\xi}_1 = \frac{\xi_1}{1-u_{o1,1}} ; \quad \bar{\xi}_2 = \frac{\xi_2}{1-u_{o2,2}} ; \quad \bar{\xi}_3 = \frac{\xi_3}{1-u_{o3,3}} , \quad (4.2.17)$$

and remembering that $u_{oi,j}=0$ for $i \neq j$, one can rewrite Equation 4.2.16 as

$$v_i = -G\bar{\xi}_i + v_n \bar{\xi}_i ; \quad v_n = v_i n_i . \quad (4.2.18)$$

Multiply the first of Equation 4.2.18 by n_i , solve for v_n and then substitute for v_n to obtain

$$[v_i] = v_i = - \frac{G\bar{\xi}_i}{1-\bar{\xi}_n} ; \quad \bar{\xi}_n = \bar{\xi}_i n_i . \quad (4.2.19)$$

From Equation 3.4.23, the density in the medium ahead is given by

$$\frac{\rho_o}{\rho_{oo}} = \{1-2I_o+4II_o-8III_o\}^{1/2} , \quad (4.2.20a)$$

$$= \{\det(\delta_{ij}-2e_{oij})\}^{1/2} , \quad (4.2.20b)$$

where I_o , II_o and III_o are invariants of e_{oij} .

However, since e_{oij} is diagonal, this reduces to

$$\frac{\rho_o}{\rho_{oo}} = \{(1-2e_{o11})(1-2e_{o22})(1-2e_{o33})\}^{1/2} . \quad (4.2.21)$$

From Equations 4.2.10, one also has

$$1-2e_{o11}=(1-u_{o1,1})^2; \quad 1-2e_{o22}=(1-u_{o2,2})^2; \quad 1-2e_{o33}=(1-u_{o3,3})^2. \quad (4.2.22)$$

Therefore Equation 4.2.21 simplifies to

$$\rho_o = \rho_{oo}(1-u_{o1,1})(1-u_{o2,2})(1-u_{o3,3}). \quad (4.2.23)$$

The jump in the strain tensor is now obtained from

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j},$$

as

$$\begin{aligned} 2[e_{ij}] &= [u_{i,j}] + [u_{j,i}] - [u_{k,j}][u_{k,i}] \\ &= [u_{i,j}] + [u_{j,i}] - [u_{k,i}][u_{k,j}] - u_{ok,i}[u_{k,j}] \\ &\quad - u_{ok,j}[u_{k,i}] \\ &= \xi_i n_j + \xi_j n_i - \xi^2 n_i n_j - \xi_k (u_{ok,i} n_i + u_{ok,j} n_j), \end{aligned} \quad (4.2.24)$$

where $\xi^2 = \xi_k \xi_k$ and the bracket definition given in Equation 2.5.20 were used.

The first shock condition, Equation 2.5.16, reduces to

$$\rho(v_n - G) = \rho_o(v_{on} - G) = -\rho_o G, \quad (4.2.25)$$

since $v_{on} = 0$.

Use of Equations 4.2.19 in Equation 4.2.25, yields an expression for the density of the shocked medium, as

$$\rho = \rho_0(1-\bar{\xi}_n); \quad [\rho] = -\rho_0\bar{\xi}_n/(1-\bar{\xi}_n). \quad (4.2.26)$$

Consider now the shock condition that follows from the balance of linear momentum, which is given in Equation 2.5.17 as

$$\rho_0(v_{on}-G)[v_i] = [t_{ji}]n_j. \quad (4.2.27)$$

From Equations 4.2.19 and 4.2.25, these reduce to

$$[f_i] = \frac{\rho_0 G^2 \bar{\xi}_i}{1-\bar{\xi}_n}; \quad f_i = t_{ji}n_j. \quad (4.2.28)$$

This is a very important result which distinguishes elastic shocks from those in gas-dynamics. Of course, it gives the common result that the shock normal speed G depends on the amplitude; it is only for weak shocks in the limit as $\xi_i \rightarrow 0$ (which yields infinitesimal waves), that it may be possible that $[f_i]$ is $O(\xi_i)$ and thus one obtains a constant speed. In general the shock speed G depends on the amplitude, as measured here by ξ_i . In gas-dynamics, $t_{ij} = -P\delta_{ij}$ and thus the tangential component of the velocity is continuous. Hence it is only the normal speed that suffers a discontinuity and this is related to the jump in density; which can be taken as the strength of the shock. The shock in gas-dynamics is thus automatically a one parameter family. But in elasticity no such unique statement can be made. However, for a given shock, the speed G given by the three Equations 4.2.28 must be identical; this imposes the con-

sistency condition

$$\frac{[f_1]}{\bar{\xi}_1} = \frac{[f_2]}{\bar{\xi}_2} = \frac{[f_3]}{\bar{\xi}_3} . \quad (4.2.29)$$

These $[f_i]$ are functions of ξ_i ; thus Equation 4.2.29 provides two relations for three unknowns ξ_i . The function $f_i = t_{ji} n_j$, which is obtained from the internal energy function, is quite complicated; so these relations are by no means simple. However, in principle, these consistency relations always enable one to convert a given shock into a single parameter (say ϵ) family which may be identified with one of ξ_i too, as is done later.

The Rankine-Hugoniot relation for this case is now obtained from the generalized Rankine-Hugoniot relation in Equation 4.2.8. First it is written in terms of $[f_i]$ as

$$[U] = \frac{-\rho_{oo} [\rho] (f_i + f_{oi}) [f_i]}{2\rho_o \rho [f_n]} ; \quad [f_n] = [f_i] n_i . \quad (4.2.30)$$

Use of the second of Equation 4.2.26 and the first of Equation 4.2.28, gives

$$[U] = \frac{\rho_{oo} \bar{\xi}_i (f_i + f_{oi})}{2\rho_o (1 - \bar{\xi}_n)} . \quad (4.2.31)$$

This depends only on the state variables U and ξ_i since the f_i depend only on the ξ_i through the constitutive law.

Another important distinction of this relation from that

in other fields must be noted. Though gas-dynamic shock waves are a single parameter family, multiple-parameter shock waves do occur in other fields besides elasticity, such as in magnetogasdynamics. In the latter case, tangential speed is not continuous. In the gas-dynamics case no geometry of the shock appears; in the magnetogasdynamics case, the generalized Rankine-Hugoniot relation [30, p. 217] contains the tangential component of the magnetic field. Hence the geometry is involved in the sense that it involves the tangent vector. The generalized Rankine-Hugoniot relation of elasticity, Equation 4.2.8, involves not only the geometry, through the appearance of n_i , but it will involve curvatures too through the stress tensor t_{ij} which depends on the strain which, in turn, will involve curvatures in the general case.

For a shock wave moving into an initially unstrained medium at rest in its natural state, one has

$$t_{oij} \equiv 0; \quad e_{oij} \equiv 0; \quad \bar{\xi}_i = \xi_i; \quad v_{oi} \equiv 0, \quad (4.2.32)$$

and Equations 4.2.19, 4.2.24, 4.2.26, 4.2.28, 4.2.29, and 4.2.31 simplify to

$$\rho_o = \rho_{oo}; \quad \rho = \rho_{oo}(1 - \xi_n); \quad v_i = - \frac{G\xi_i}{1 - \xi_n}, \quad (4.2.33)$$

$$2e_{ij} = \xi_i n_j + \xi_j n_i - \xi^2 n_i n_j; \quad \xi^2 = \xi_k \xi_k, \quad (4.2.34)$$

$$f_i = t_{ji} n_j = \frac{\rho_{\infty} G^2 \xi_i}{1 - \xi_n} ; \quad \frac{f_1}{\xi_1} = \frac{f_2}{\xi_2} = \frac{f_3}{\xi_3} , \quad (4.2.35)$$

$$[U] = \frac{f_i \xi_i}{2(1 - \xi_n)} ; \quad \xi_n = \xi_i n_i . \quad (4.2.36)$$

4.3. General Classification of Shock Waves

It is a well-known result that in isotropic linear elasticity there is one dilatational, longitudinal wave and two equivoluminal, transverse waves. This common feature is isotropic linear elasticity has sometimes caused confusion in the use of terminology. A longitudinal wave is here understood as the one whose amplitude vector is perpendicular to the wave-front; whereas a dilatational wave is one accompanied by density changes. Further throughout this study the amplitude of the shock wave is measured by the vector ξ_i , which is related to the jump in the displacement gradients by $[u_{i,j}] n_j = \xi_i$; if n_i is the unit normal to the shock-front and t_i and s_i are the unit tangent vectors, with

$$\xi_i = \xi_n n_i + \xi_t t_i + \xi_s s_i, \quad (4.3.1)$$

then $\xi_n \neq 0$, $\xi_t = \xi_s = 0$ is pure longitudinal. $\xi_n = 0$ and $\xi_s = 0$, $\xi_t \neq 0$ or $\xi_t = 0$, $\xi_s \neq 0$ makes the wave purely transverse. Finally $\xi_n \neq 0$, $\xi_t \neq 0$ and $\xi_n = o(\xi_t)$ as $\xi_t \rightarrow 0$ makes the wave mixed transverse-

longitudinal and $[\rho] \neq 0$ gives the dilatational character. As will be found in the discussion below, there is no wave in a compressible elastic medium which is unaccompanied with density changes. The notion of a shock wave, in the sense used in the present study, simply means that the discontinuity is in the derivatives of the displacement gradients. Further the characterization of a purely transverse wave also loses meaning. In the results that follow it is shown that whether the medium ahead is strained or not, though one can have pure longitudinal shocks, there do not exist pure transverse shocks at all. In order to bring out these results, plane shocks in an initially unstrained medium are discussed and then considerations generalized to arbitrary shock waves travelling in an initially unstrained medium. Also some results for plane shock waves travelling in an initially unstrained anisotropic medium at rest are obtained in a separate chapter.

4.4. Plane Pure Longitudinal Shock Waves

For plane shocks in general, choose $n_i = (1, 0, 0)$ and $\xi_i = (\xi, \eta, \zeta)$. Further since it is initially unstrained and at rest in its natural state, one has

$$\begin{aligned} \rho_{00} &= \rho_0; \quad [u_{i,j}] = u_{i,j}; \quad [v_i] = v_i, \\ [e_{ij}] &= e_{ij}; \quad [t_{ij}] = t_{ij}. \end{aligned} \tag{4.4.1}$$

The first and second shock conditions, the consistency condition and the generalized Rankine-Hugoniot relation, given Equations 4.2.33, 4.2.34, 4.2.35 and 4.2.36, are respectively

$$\rho = \rho_{oo}(1-\xi); \quad v_i = -\frac{G\xi_i}{1-\xi}; \quad t_{11} = \frac{\rho_{oo}G^2\xi}{1-\xi}, \quad (4.4.2)$$

$$t_{12} = \frac{\rho_{oo}G^2\eta}{1-\xi}; \quad t_{13} = \frac{\rho_{oo}G^2\zeta}{1-\xi}; \quad \frac{t_{11}}{\xi} = \frac{t_{12}}{\eta} = \frac{t_{13}}{\zeta}, \quad (4.4.3)$$

$$[U] = U - U_{oo} = \frac{f_i \xi_i}{2(1-\xi)}. \quad (4.4.4)$$

The strain tensor is given by

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & 0 & 0 \\ e_{13} & 0 & 0 \end{pmatrix}; \quad e^2 = \begin{pmatrix} e_{11}^2 + e_{12}^2 + e_{13}^2 & e_{11}e_{12} & e_{11}e_{13} \\ e_{11}e_{12} & e_{12}^2 & e_{11}e_{23} \\ e_{11}e_{13} & e_{11}e_{23} & e_{13}^2 \end{pmatrix}, \quad (4.4.5)$$

with

$$e_{11} = \xi - \frac{1}{2}(\xi^2 + \eta^2 + \zeta^2); \quad e_{12} = \frac{1}{2}\eta; \quad e_{13} = \frac{1}{2}\zeta. \quad (4.4.6)$$

Similarly, from Equations 3.3.2, 3.3.3, 3.3.4, 3.3.5, 3.4.22 and 4.4.5, the stress tensor and the strain invariants reduce to

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & 0 \\ t_{13} & 0 & t_{33} \end{pmatrix}, \quad (4.4.7)$$

with

$$\begin{aligned} t_{11} &= a + be_{11} + c\{e_{11}^2 + e_{12}^2 + e_{13}^2\} \\ &= \frac{\rho}{\rho_{00}} \{U_1(1 - 2e_{11}) + 2U_2(e_{12}^2 + e_{13}^2)\}, \end{aligned} \quad (4.4.8)$$

$$t_{12} = (b + ce_{11})e_{12} = -\frac{\rho}{\rho_{00}} e_{12}(2U_1 + U_2), \quad (4.4.9)$$

$$t_{13} = (b + ce_{11})e_{13} = -\frac{\rho}{\rho_{00}} e_{13}(2U_1 + U_2), \quad (4.4.10)$$

$$t_{22} = a + ce_{12}^2 = \frac{\rho}{\rho_{00}} \{U_1 + U_2(e_{11} + 2e_{12}^2) - U_3e_{13}^2\}, \quad (4.4.11)$$

$$t_{33} = a + ce_{13}^2 = \frac{\rho}{\rho_{00}} \{U_1 + U_2(e_{11} + 2e_{13}^2) - U_3e_{12}^2\}, \quad (4.4.12)$$

$$I = e_{11} \quad II = -e_{12}^2 - e_{13}^2; \quad III = 0, \quad (4.4.13)$$

where a , b and c are given in Equation 3.4.21 and

$$U_1 = U_1(I, II, S) = \left. \frac{\partial U}{\partial I} \right|_{S, II=\text{constant}}, \text{ etc.} \quad (4.4.14)$$

Further note that though $III=0$, U_3 may not be zero. However it only affects t_{22} and t_{33} , which do not enter into further discussion in this study.

It has to be remembered that stress is assumed to

vanish with strain.

From these equations, it can be seen that it is admissible to take $\eta=\zeta=0$ which will reduce the first and second forms of Equation 4.4.3 to identities and the last form in Equation 4.4.2 is again

$$t_{11} = \frac{\rho_{00} G^2 \xi}{1-\xi} . \quad (4.4.15)$$

Equations 4.4.5-4.4.13 reduce to

$$e_{ij} = \begin{pmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad e^2 = \begin{pmatrix} e_{11}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad (4.4.16)$$

$$t_{ij} = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix} ; \quad e_{11} = \xi - \frac{1}{2}\xi^2 , \quad (4.4.17)$$

$$t_{11} = \frac{\rho}{\rho_{00}} U_1 (1-2e_{11}) ; \quad t_{22}=t_{33} = \frac{\rho}{\rho_{00}} \{U_1 + U_2 e_{11}\} , \quad (4.4.18)$$

$$I = e_{11} ; \quad II=III=0. \quad (4.4.19)$$

Thus a pure dilatational, longitudinal shock is possible in an initially unstrained elastic medium. Its longitudinal character is given by $\xi_i n_i \neq 0$ and the dilatational character is given by $[\rho] = -\rho_{00} \xi \neq 0$. The normal speed of

propagation G is clearly dependent on the amplitude unless $t_{11}(\xi, S)$ turns out to be exactly proportional to $\{\xi/(1-\xi)\}$, which it does not. Further $\xi < 0$ gives a compressive shock with the shocked medium increasing in density and $\xi > 0$ gives a rarefaction shock. Since density cannot vanish, from the first of Equation 4.4.2, $-\infty < \xi < 1$; the density relation thus providing no lower bound to ξ .

In gas-dynamics one obtains an upper bound for density increase. Such a conclusion is difficult to infer in elasticity due to lack of knowledge of the internal energy function. Indeed this criteria may provide a constraint on the allowable forms of the internal energy function.

Consider now entropy changes assuming that ξ is small but still finite in order to justify conclusions from Taylor series expansions; note that ξ need not be infinitesimal (which leads to the linear theory).

The Rankine-Hugoniot relation given in Equation 4.4.4 reduces to the form

$$[U] = U - U_{\infty} = \frac{t_{11}\xi}{2(1-\xi)} . \quad (4.4.20)$$

Gibb's equation, given in Equation 3.5.30, reduces to the form

$$dU = \rho_{\infty} \theta dS + \frac{\rho_{\infty}}{\rho} \left\{ \frac{t_1 de_1}{1-2e_1} \right\}$$

$$\begin{aligned}
&= \rho_{oo} \theta ds + \frac{\rho_{oo}}{\rho} \left\{ \frac{t_{11}(1-\xi) d\xi}{(1-\xi)^2} \right\} \\
&= \rho_{oo} \theta ds + \frac{t_{11} d\xi}{(1-\xi)^2} , \tag{4.4.21}
\end{aligned}$$

Note that for this case one has $t_{11}=t_1$, $e_{11}=e_1=\xi-\frac{1}{2}\xi^2$ and $e_2=e_3=0$; further the first of Equation 4.4.2, $\rho=\rho_{oo}(1-\xi)$, was used above.

Now in the present case, $U=U(\xi, S)$ and $t_{11}=t_{11}(\xi, S)$; further Equation 4.4.20 provides an additional relation between these. So one can regard all functions, including S , as functions of a single variable ξ . Thus the shock is reduced to a one parameter family, the parameter being now the amplitude ξ , which can be taken to be the strength of the shock.

Once the shock is reduced to a one parameter family, the following interpretation of the total derivative with respect to the single shock parameter (which in this study will always be the strength of the shock), denoted by a dot throughout this study, is to be understood. For any function, say $g(\xi, S)$, one writes

$$\dot{g} = \frac{\partial g}{\partial \xi} + \frac{\partial g}{\partial S} \dot{S} = g_{\xi} + g_S \dot{S} ,$$

with the partial derivatives having the usual meaning.

Differentiating Equations 4.4.20 and 4.4.21 with respect to ξ , one obtains respectively

$$2\dot{U} = \frac{\dot{t}_{11}\xi}{1-\xi} + \frac{t_{11}}{1-\xi} + \frac{t_{11}\xi}{(1-\xi)^2}, \quad (4.4.22)$$

$$2\dot{U} = 2\rho_{oo}\theta\dot{S} + \frac{2t_{11}}{(1-\xi)^2}. \quad (4.4.23)$$

Equating values of \dot{U} from Equations 4.4.22 and 4.4.23, one obtains

$$2\rho_{oo}\theta\dot{S} = \frac{\dot{t}_{11}\xi}{1-\xi} - \frac{t_{11}}{(1-\xi)^2}. \quad (4.4.24)$$

Now since an unstrained material is unstressed, then

$$t_{11}(\xi, S) \bigg|_{\substack{\xi=0 \\ S=S_{oo}}} = t_{11}(0, S_{oo}) = t_{o11} = 0. \quad (4.4.25)$$

So evaluating Equation 4.9.24 for $\xi=0$ yields

$$\dot{S} \bigg|_{\xi=0} = \dot{S}_o = 0; \quad S=S(\xi); \quad (\rho_{oo}, \theta_o \neq 0), \quad (4.4.26)$$

where the suffix "o" means evaluation at the state $\xi=0$ and $S(0)=S_{oo}$.

Differentiate Equation 4.4.24 with respect to ξ again, to obtain

$$2\rho_{oo}(\dot{\theta}\dot{S} + \theta\ddot{S}) = \frac{\ddot{t}_{11}}{1-\xi} - \frac{2t_{11}}{(1-\xi)^3}. \quad (4.4.27)$$

Evaluate Equation 4.4.27 now for $\xi=0$, $S=S_{oo}$; since \dot{S} and t_{11} vanish for $\xi=0$, this gives

$$\ddot{S}_o = 0, \quad (\rho_{oo}, \theta_o \neq 0). \quad (4.4.28)$$

Another differentiation of Equation 4.4.27 with respect to ξ yields

$$2\rho_{oo}(\ddot{\theta}\dot{S}+2\dot{\theta}\ddot{S}+\theta\ddot{\ddot{S}}) = \frac{\ddot{t}_{11}\xi}{1-\xi} + \frac{\ddot{t}_{11}}{1-\xi} - \frac{2\dot{t}_{11}}{(1-\xi)^2} - \frac{6t_{11}}{(1-\xi)^4}. \quad (4.4.29)$$

Evaluate Equation 4.4.29 for $\xi=0$, $S=S_{oo}$; since \dot{S} , \ddot{S} and t_{11} vanish for $\xi=0$, one obtains the final result that

$$2\rho_{oo}\theta_o\ddot{\ddot{S}}_o = \{\ddot{t}_{11}-2\dot{t}_{11}\}_o. \quad (4.4.30)$$

Note further the interpretation of the dot derivatives as

$$\begin{aligned} \dot{t}_{11} &= \frac{\partial t_{11}}{\partial \xi} + \frac{\partial t_{11}}{\partial S} \dot{S} ; \quad \ddot{t}_{11} = \frac{\partial^2 t_{11}}{\partial \xi^2} + 2 \frac{\partial^2 t_{11}}{\partial \xi \partial S} \dot{S} \\ &+ \frac{\partial^2 t_{11}}{\partial S^2} \dot{S}^2 + \frac{\partial t_{11}}{\partial S} \ddot{S} , \end{aligned} \quad (4.4.31)$$

where in the partial differentiation of $t_{11}(\xi, S)$, the usual meaning of the partial derivative is to be understood, e.g., $\partial t_{11}/\partial \xi$ denoting the partial derivative with respect to ξ with S held constant.

However when the total derivatives in Equation 4.4.31 are evaluated at $\xi=0$, the terms involving the partial derivatives with respect to S drop out since \dot{S} and \ddot{S} vanish with ξ . Thus Equation 4.4.30 reduces to

$$2\rho_{oo}\theta_o\ddot{\ddot{S}}_o = \left(\frac{\partial^2 t_{11}}{\partial \xi^2} - 2 \frac{\partial t_{11}}{\partial \xi} \right)_{\substack{\xi=0 \\ S=S_{oo}}} . \quad (4.4.32)$$

Equation 4.4.32 can be reduced to a very interesting form. Truesdell [66] obtains the square of the speed of propagation of acceleration waves in terms of the first derivative of tension with respect to stretch (which of course gives that this derivative must be positive since the square is always positive). The result given in Equation 4.4.32 can now be reduced to the second derivative of the tension, t_{11} , with respect to stretch. It is later shown from Equation 4.4.32 and an additional physical requirement, that the second derivative must also be positive. To obtain this consider the homogeneous deformation given by

$$x_a = \lambda_a X_a, \quad (a \text{ not summed; } a=1,2,3), \quad (4.4.33)$$

where the λ_a are the principal stretches. Then since

$$u_a = \left\{1 - \frac{1}{\lambda_a}\right\} x_a, \quad (4.4.33a)$$

one has

$$1-2e_a = \frac{1}{\lambda_a^2}, \quad a=1,2,3. \quad (4.4.34)$$

For this problem, using the second of Equation 4.4.17 with Equation 4.3.34, one thus has

$$1-\xi = \frac{1}{\lambda_1}, \quad (4.4.34a)$$

and

$$\frac{\partial t_{11}}{\partial \xi} = \frac{1}{\lambda_1^2} \frac{\partial t_{11}}{\partial \lambda_1}, \quad (4.4.34b)$$

$$\frac{\partial^2 t_{11}}{\partial \xi^2} = \frac{1}{\lambda_1^4} \frac{\partial^2 t_{11}}{\partial \lambda_1^2} + \frac{2}{\lambda_1^3} \frac{\partial t_{11}}{\partial \lambda_1} . \quad (4.4.34c)$$

Substituting Equations 4.4.34b and 4.4.34c into Equation 4.4.32 and then evaluating at $\xi=0$ and $S=S_{oo}$ one obtains the result

$$2\rho_{oo} \ddot{S}_o = \left. \frac{\partial^2 t_{11}}{\partial \lambda_1^2} \right|_{\substack{\xi=0 \\ S=S_{oo}}} . \quad (4.4.35)$$

Thus while the tangent to the tension-stretch curve gives the speed of propagation of acceleration waves, its curvature gives the entropy change across a weak shock.

For later reference, Equation 4.4.32 will be expressed in terms of derivatives of the internal energy function. To this end, from the first of Equation 4.4.18, one has

$$t_{11} = \frac{\rho}{\rho_{oo}} U_1 (1-2e_{11}) = (1-\xi)^3 U_1 . \quad (4.4.36)$$

Differentiating Equation 4.4.36 twice with respect to ξ and evaluating for $\xi=0$, yields

$$\dot{t}_{11} = (1-\xi)^3 \dot{U}_1 - 3(1-\xi)^2 U_1 , \quad (4.4.37)$$

$$\ddot{t}_{11} = (1-\xi)^3 \ddot{U}_1 - 6(1-\xi)^2 \dot{U}_1 + 6(1-\xi) U_1 , \quad (4.4.38)$$

$$\dot{t}_{11})_o = \dot{U}_1)_o , \quad (4.4.39)$$

$$\ddot{t}_{11})_o = \{\ddot{U}_1 - 6\dot{U}_1\}_o , \quad (4.4.40)$$

where it was used that, since stress vanishes with strain, $U_1)_0 = 0$.

Further the total derivatives of the internal energy function can be expressed in terms of partial derivatives with respect to ξ as

$$\dot{U}_1 = U_{11}\dot{I} + U_{1S}\dot{S} ; \quad II=III=0, \quad (4.4.41)$$

$$\ddot{U}_1 = U_{111}\dot{I}^2 + U_{11}\ddot{I} + 2U_{11S}\dot{I}\dot{S} + U_{1SS}\dot{S}^2 + U_{1S}\ddot{S} . \quad (4.4.42)$$

Evaluating at $\xi=0$ yields

$$\dot{U}_1)_0 = U_{11})_0 ; \quad \ddot{U}_1)_0 = \{U_{111}-U_{11}\}_0, \quad (4.4.43)$$

where it was used that $\dot{S}_0=\ddot{S}_0=0$ and from Equation 4.4.13

that

$$\dot{I} = \dot{\epsilon}_{11} = 1-\xi ; \quad \dot{I}_0 = 1, \quad (4.4.44)$$

$$\ddot{I} = -1 ; \quad \ddot{I}_0 = -1 . \quad (4.4.45)$$

Combining Equations 4.4.32, 4.4.39, 4.4.40, and 4.4.43, one finally obtains

$$2\rho_{00}\theta_0\ddot{S}_0 = \{U_{111}-9U_{11}\}_0 . \quad (4.4.46)$$

Consider now the arguments needed to clarify the remarks made in the paragraph just before Equation 4.4.20. Since due to thermodynamical considerations and the existence of the Rankine-Hugoniot relation, one can regard S as a function of ξ only, a Taylor series expansion about $\xi=0$, $S=S_{00}$ can be

assumed as

$$[S]=S-S_{OO}=\dot{S}_O \xi + \frac{1}{2} \ddot{S}_O \xi^2 + \frac{1}{6} \dddot{S}_O \xi^3 + \frac{1}{24} S^{(4)}_O \xi^4 + \dots \quad (4.4.47)$$

where the dot means the derivative with respect to ξ and the suffix "o" again means evaluated at $\xi=\xi_O=0$ and $S=S_{OO}=S_O$.

The assumption of the existence of a Taylor series may not itself be regarded as quite a strong restriction; for such a series the radius of convergence may be quite large. However the main problem is if one allows ξ to be quite large then this expansion is inadequate for the forthcoming physical conclusion to be drawn in full generality. Thus a further constraint is imposed that ξ is small enough but finite in order to approximate the series given in Equation 4.4.47 only by the first non-vanishing term. This is a rather unfortunate restriction which is removable in gas-dynamics. It is felt that a better knowledge of state functions in elasticity may enable one to extend analogous arguments in elasticity too. However this will not be done here.

Making the assumption explained in the above paragraph, one can now assert that entropy changes are of third-order in the shock-strength, as measured here by ξ .

Further one imposes the basic physical requirement that entropy must increase. Based on the stated approximation above, this implies, from Equations 4.4.32, 4.4.35 and 4.4.47, that

$$\left(\frac{\partial^2 t_{11}}{\partial \xi^2} - 2 \frac{\partial t_{11}}{\partial \xi} \right)_{\substack{\xi=0 \\ S=S_{00}}} \xi^3 > 0, \quad (4.4.48)$$

or

$$\left(\frac{\partial^2 t_{11}}{\partial \lambda_1^2} \right)_{\substack{\xi=0 \\ S=S_{00}}} \xi^3 > 0. \quad (4.4.49)$$

Since $\rho = \rho_0(1-\xi)$, $\xi < 0$ gives the compressive shock waves.

Thus one has the criteria for a compressive shock as

$$\left\{ \frac{\partial^2 t_{11}}{\partial \xi^2} - 2 \frac{\partial t_{11}}{\partial \xi} \right\}_0 < 0, \quad (4.4.50)$$

or

$$\left(\frac{\partial^2 t_{11}}{\partial \lambda_1^2} \right)_0 < 0. \quad (4.4.51)$$

Since $e_{11} = \xi - \frac{1}{2}\xi^2$, Equation 4.4.50 can be written as

$$\left(\frac{\partial^2 t_{11}}{\partial e_{11}^2} - 3 \frac{\partial t_{11}}{\partial e_{11}} \right)_0 < 0. \quad (4.4.52)$$

Of course for a rarefaction shock, $\xi > 0$ and these inequalities are reversed.

Using the polynomial expansion (with only the needed terms included), one can write the internal energy as

$$U = \frac{\lambda + 2\mu}{2} I^2 - 2\mu II + \ell I^3 + mI II + nIII + pII^2 + \dots \quad (4.4.53)$$

Using Equations 4.4.26, 4.4.28, 4.4.46 and 4.4.47, one

can write the entropy change as

$$2\rho_{00}\theta_0(S-S_{00}) = \frac{1}{6}\{6\ell-9(\lambda+2\mu)\}\xi^3+0(\xi^4) . \quad (4.4.54)$$

Then one has, from Equation 4.4.50, that for a compressive shock

$$\ell - \frac{3}{2}(\lambda+2\mu) < 0, \quad (4.4.55)$$

or

$$\frac{\ell}{\mu} - \frac{3}{2}\left(\frac{\lambda}{\mu}\right) - 3 < 0 . \quad (4.4.56)$$

A known value of ℓ and m [67] may be worth noting; in terms of the present formulation they reduce to (see Appendix A)

$$\frac{\ell}{\mu} = -7.28; \quad \frac{m}{\mu} = 2.64; \quad \frac{\lambda}{\mu} = 1.39 . \quad (4.4.57)$$

Thus, assuming (ℓ/μ) is negative, as in Equation 4.4.57, from Equation 4.4.54 plane pure longitudinal shock waves in an isotropic hyperelastic medium, initially unstrained and at rest, must be compressive; which is the case in gas-dynamics.

4.5. Plane Shock Waves of the Mixed Transverse-Longitudinal Type

The type that is discussed here is an extremely interesting and complicated one. It is an example showing that the linear analysis can be quite misleading. Since the medium

is initially unstrained and at rest in its natural state, one again has

$$\begin{aligned} \rho_{00} &= \rho_0; [u_{i,j}] = u_{i,j}; [v_i] = v_i, \\ [e_{ij}] &= e_{ij}; [t_{ij}] = t_{ij}. \end{aligned} \quad (4.5.1)$$

Further with the plane of the shock again taken as normal to the x_1 -axis and choosing $n_i = (1, 0, 0)$, the discontinuities in the displacement gradients are given by

$$[u_{i,1}] = u_{i,1} = \xi_i = (\xi, \eta, \zeta), \quad (4.5.2)$$

or

$$u_{1,1} = \xi; u_{2,1} = \eta; u_{3,1} = \zeta, \quad (4.5.3)$$

and the rest of the $u_{i,j}$ being zero. Thus, from Equation 4.4.2 and the first and second of Equation 4.4.3, the general plane shock is given by

$$\begin{aligned} \rho &= \rho_{00}(1-\xi); v_i = -\frac{G\xi_i}{1-\xi}; \frac{\rho_{00}G^2\xi}{1-\xi} = t_{11}, \\ \frac{\rho_{00}G^2\eta}{1-\xi} &= t_{12}; \frac{\rho_{00}G^2\zeta}{1-\xi} = t_{13}. \end{aligned} \quad (4.5.4)$$

However a certain rotation of the 2-3 axes always exists to secure ξ_i in the form $(\xi, \eta, 0)$ in the new rotated coordinate system. So without loss of generality, from Equation 4.4.4 and 4.5.4, one can take

$$\rho = \rho_{00}(1-\xi) ; \frac{\rho_{00}G^2\xi}{1-\xi} = t_{11} ; \frac{\rho_{00}G^2\eta}{1-\xi} = t_{12}, \quad (4.5.5)$$

$$\frac{t_{11}}{\xi} = \frac{t_{12}}{\eta} ; [U] = \frac{t_{11}\xi + t_{12}\eta}{2(1-\xi)} . \quad (4.5.6)$$

Further from Equations 4.4.5-4.4.13, one has for this case

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad e^2 = \begin{pmatrix} e_{11}^2 + e_{12}^2 & e_{11}e_{12} & 0 \\ e_{11}e_{12} & e_{12}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.5.7)$$

$$e_{11} = \xi - \frac{1}{2}(\xi^2 + \eta^2); \quad e_{12} = \frac{1}{2}\eta, \quad (4.5.8)$$

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (4.5.9)$$

$$t_{11} = \frac{\rho}{\rho_{00}} \{U_1(1-2e_{11}) + 2U_2e_{12}^2\}; \quad t_{12} = \frac{-\rho}{\rho_{00}} e_{12}(2U_1 + U_2), \quad (4.5.10)$$

$$t_{22} = \frac{\rho}{\rho_{00}} \{U_1 + U_2(e_{11} + 2e_{12}^2)\}; \quad t_{33} = \frac{\rho}{\rho_{00}} \{U_1 + U_2e_{11} - U_3e_{12}^2\}, \quad (4.5.11)$$

$$I = e_{11}; \quad II = -e_{12}^2; \quad III = 0, \quad (4.5.12)$$

with

$$U_1 = U_1(I, II, S) = \left. \frac{\partial U}{\partial I} \right|_{II, S = \text{constant}}, \text{ etc.} \quad (4.5.13)$$

Firstly note that from the form of the expressions above, $t_{11}=t_{11}(\xi, \eta, S)$ and $t_{12}=t_{12}(\xi, \eta, S)$, it is clear that one cannot take as before $t_{11}(0, \eta, S) \equiv 0$; if this were possible then one could omit the result of one component of the equations of motion (the second of Equation 4.5.5) and obtain a pure shear shock with $\eta \neq 0, \xi = 0$, with no added consistency condition (the first of Equation 4.5.6) imposed, as was done in the earlier case; but this is not possible. This shows that though a pure longitudinal shock can exist, a pure transverse (shear) shock cannot exist at all. (All these conclusions are of course valid for a compressible hyperelastic medium and do not apply in the incompressible case. The incompressible medium will be discussed in separate sections and labeled as such; in all other sections a compressible medium is assumed.) The transverse shock, which one can expect to turn out to be the pure shear wave in linear theory, is always accompanied with a longitudinal component in the nonlinear theory. This is the situation which gives cross-effects (Kelvin-Pointing effects) in nonlinear elasticity. These effects, known to the literature only through static problems, imply the existence of (t_{11}, t_{22}, t_{33}) in addition to t_{12} for a pure shear strain given by $e_{12} \neq 0$ and $e_{ij} = 0$ otherwise. The

dynamical problem says more; since $\xi \neq 0$ there is not only the presence of (t_{11}, t_{22}, t_{33}) but also e_{11} too. Further these effects are said to be second-order, a term which does not appear to be clear in wave motion. This last statement will be made more precise in what follows.

Write first the consistency condition given in the first of Equation 4.5.6 as

$$\frac{t_{11}}{\xi} = \frac{t_{12}}{\eta} = f(\xi, \eta, S). \quad (4.5.14)$$

Note that $f(\xi, \eta, S)$ has a finite value as $\eta \rightarrow 0$.

Consider a rotation of 180° about the x_1 -axis. The transformation matrix Q is given by

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.5.15)$$

Under this transformation, the stress (t_{ij}) and the strain (e_{ij}) tensors and the discontinuity vector ξ_i of the present case transform as

$$\begin{aligned} Q t Q^t &= \begin{pmatrix} t_{11} & -t_{12} & 0 \\ -t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}; & Q e Q^t &= \begin{pmatrix} e_{11} & -e_{12} & 0 \\ -e_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ Q \xi &= (\xi, -\eta, 0). \end{aligned} \quad (4.5.16)$$

Thus the consistency condition, Equation 4.5.14, given by

$$\frac{t_{11}(\xi, \eta, S)}{\xi} = \frac{t_{12}(\xi, \eta, S)}{\eta} = f(\xi, \eta, S),$$

transforms to

$$\frac{t_{11}(\xi, -\eta, S)}{\xi} = \frac{t_{12}(\xi, -\eta, S)}{\eta} = f(\xi, -\eta, S). \quad (4.5.17)$$

Thus if the consistency condition determines ξ as a function of η , $\xi(\eta, S)$, then $\xi(-\eta, S)$ is also a solution. This implies that ξ is an even function of η .

This result of applying the transformation Q is physically obvious. For example a shear in the 2-3 plane parallel to the 2-axis causes a change in density; this change in density should not depend on whether the shear is in the +2-direction or in the -2-direction. The same can be said about invariance of changes of sign for ξ , t_{11} , t_{22} and t_{33} ; it is only t_{12} that should change sign. Thus ρ , ξ , t_{11} , t_{22} and t_{33} are all even in η while t_{12} is an odd function of η . This is what the above transformation achieved.

The information that ξ is an even function of η and always accompanies any value of η does not suffice to characterize the shock wave completely. ξ can be an even function of η but larger than η . Then the shock wave has to be characterized only as predominantly longitudinal but still accompanied by a weaker transverse component. This mixed

longitudinal-transverse shock wave is also a mixed dilational-shear shock wave. This mixed type with $\xi > \eta$ has no analogue at all in linear theory.

However imposing one more constraint does give the wave of linear theory. Assume $\xi = o(\eta)$ and ξ vanishes with η and then it is reasonable to assume

$$\xi = \sum_{n=0}^{\infty} k_n \eta^{2n+2} . \quad (4.5.18)$$

The constants k_n 's are to be determined from the consistency condition, Equation 4.5.14. It has to be remembered that the k_n 's are functions of entropy S . Since η is small but finite and $\xi = o(\eta)$, it is reasonable to assume an expansion for the functions in Equation 4.5.14 and equate coefficients of like powers of η . To this end first obtain the consistency condition, Equation 4.5.14, in terms of derivatives of the internal energy function U . Using Equation 4.5.10 in Equation 4.5.14 yields

$$\frac{(1-\xi)^2 U_1 + \eta^2 (U_1 + \frac{1}{2} U_2)}{\xi} = -U_1 - \frac{1}{2} U_2 . \quad (4.5.19a)$$

This exhibits clearly that ξ is an even function of η .

Using the polynomial expansion for U given in Equation 4.4.53, Equation 4.5.19a becomes

$$\begin{aligned}
& \frac{(1-2\xi+\xi^2)}{\xi} \{ (\lambda+2\mu) (\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2) + m(-\frac{1}{4}\eta^2) + 3\ell(\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2)^2 + 0(\xi^3, \eta^4) \\
& + \eta^2 \{ (\lambda+2\mu) (\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2) + m(-\frac{1}{4}\eta^2) + 3\ell(\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2)^2 + 0(\xi^3, \eta^4) \\
& + \mu + \frac{1}{2}m(\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2) + 0(\xi^2, \eta^2) \} \} = - \{ (\lambda+2\mu) (\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2) \\
& + m(-\frac{1}{4}\eta^2) + 3\ell(\xi - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2)^2 + 0(\xi^3, \eta^4) + \mu + \frac{1}{2}m(-\frac{1}{4}\eta^2) + 0(\xi^3, \eta^2) \}, \\
& \hspace{25em} (4.5.19b)
\end{aligned}$$

which reduces to

$$\lambda + 2\mu - \left\{ \frac{m+2\lambda+8\mu}{4} \right\} \frac{\eta^2}{\xi} + 0\left(\frac{\eta^4}{\xi}, \xi\right) = \mu + 0(\eta^2, \xi). \quad (4.5.20)$$

Assuming $\xi = o(\eta)$ and ξ vanishes with η and further since η is assumed small but finite, the expansion in Equation 4.5.18 can be written as

$$\xi = \xi(\eta, S) = k(S)\eta^2 + 0(\eta^4). \quad (4.5.21)$$

Substituting Equation 4.5.21 into Equation 4.5.20 leads to the relation

$$k = \frac{m+2\lambda+8\mu}{4(\lambda+\mu)}. \quad (4.5.22)$$

The assumption that $\eta = o(\xi)$ and η vanishes with ξ ($\xi > \eta$ case) leads, from Equation 4.5.20, to the relation $\lambda + \mu = 0$. However, if the polynomial expansion is assumed to lead to the linear problem in the limit, then λ and μ must be

positive and so this is regarded as unphysical. This result and thus this wave type can also be argued as inadmissible in the following way: If one considers the difference between the longitudinal wave speed squared and the shear wave speed squared in the linear theory, it yields $\lambda + \mu$, but since these speeds are squared then $\lambda + \mu > 0$. Therefore $\lambda + \mu \neq 0$, and thus a shock with dominant longitudinal mode but accompanied by a weaker transverse mode appears to be impossible. However, this conclusion is of limited validity, since it is based only on the use of polynomial expansions. This question can only be settled by looking more exhaustively at solutions of Equation 4.5.19a, which is quite difficult with the present state of lack of knowledge of the function U .

Thus the mixed transverse-longitudinal shock wave, i.e., a dominant transverse (shear) mode accompanied by a weaker longitudinal (dilatational) mode, and the pure longitudinal shock wave are the only admissible plane shock waves in an initially unstrained compressible hyperelastic medium at rest. This mixed transverse-longitudinal shock wave is also a mixed shear-dilatational shock wave.

In the analysis that follows the principal strains will be needed. The equation which determines the principal values is

$$|e_{ij} - e\delta_{ij}| = 0. \quad (4.5.23)$$

For this case it reduces to

$$e^3 - e_{11}e^2 - e_{12}^2e = 0 \quad (4.5.24)$$

The roots are

$$\begin{aligned} 2e_1 &= e_{11} + (e_{11}^2 + 4e_{12}^2)^{1/2}; \quad 2e_2 = e_{11} - (e_{11}^2 + 4e_{12}^2)^{1/2}, \\ e_3 &= 0. \end{aligned} \quad (4.5.25)$$

The principal directions obtained from $e_{ij}l_j = el_i$ are

$$\begin{aligned} l_i^1 &= \left(\frac{2e_{12}}{(2Q + 2e_{11}\sqrt{Q})^{1/2}}, \frac{e_{11} + \sqrt{Q}}{(2Q + 2e_{11}\sqrt{Q})^{1/2}}, 0 \right); \\ l_i^2 &= \left(\frac{2e_{12}}{(2Q - 2e_{11}\sqrt{Q})^{1/2}}, \frac{e_{11} - \sqrt{Q}}{(2Q - 2e_{11}\sqrt{Q})^{1/2}}, 0 \right); \\ l_i^3 &= (0, 0, 1), \end{aligned} \quad (4.5.26)$$

with

$$Q = e_{11}^2 + 4e_{12}^2. \quad (4.5.27)$$

The Rankine-Hugoniot relation given in the last of Equation 4.5.6 can be written in terms of derivatives of U.

Using Equation 4.5.14 and the second of Equation 4.5.10,

Equation 4.5.6 becomes

$$[U] = \frac{(\xi^2 + \eta^2)f}{2(1-\xi)}, \quad (4.5.28)$$

$$= - \frac{(\xi^2 + \eta^2)(U_1 + \frac{1}{2}U_2)}{2}. \quad (4.5.29)$$

Finally the Gibb's equation, given in Equation 3.5.30, reduces in this case to

$$dU = \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho} \left\{ \frac{t_1 de_1}{1-2e_1} + \frac{t_2 de_2}{1-2e_2} \right\} , \quad (4.5.30)$$

$$= \rho_{oo} \theta dS + \frac{1}{1-\xi} \left\{ \frac{t_1 de_1}{1-2e_1} + \frac{t_2 de_2}{1-2e_2} \right\} , \quad (4.5.31)$$

where the first of Equation 4.5.5 was used.

To further reduce the Gibb's equation to a form in terms of the internal energy function U , one first obtains the principal stresses in terms of U . Using Equations 3.5.28 one can write

$$t_1 = (1-\xi)(1-2e_1)\{U_1 + e_2 U_2\} , \quad (4.5.32)$$

$$t_2 = (1-\xi)(1-2e_2)\{U_1 + e_1 U_2\} , \quad (4.5.33)$$

$$I = e_1 + e_2; \quad II = e_1 e_2; \quad III = 0. \quad (4.5.34)$$

It is to be stressed that the functional relationship given in Equation 4.5.14 gives a relationship between ξ and η taken as $\xi = \xi(\eta, S)$. So the shock is now a single parameter family; η taken as the measure of the strength of the shock, implying the case of dominant transverse mode and a weaker longitudinal mode is the one that is being studied. Thus not only $\xi = \xi(\eta, S)$, but $e_1 = e_1(\xi(\eta, S), \eta)$, $e_2 = e_2(\xi(\eta, S), \eta)$, $f = f(\xi(\eta, S), \eta, S)$ and the same holds for t_1 and t_2 . Thus since $S = S(\eta)$ too, the total derivative, denoted by a dot again

implies, e.g.,

$$\dot{f}(\xi, \eta, S) = \frac{\partial f}{\partial \xi}(\xi_{\eta} + \xi_S \dot{S}) + \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial S} \dot{S} , \quad (4.5.35)$$

where the partial derivatives have the usual meaning.

Using Equations 4.5.32 and 4.5.33, Gibb's equation, Equation 4.5.31 can be written as

$$\begin{aligned} 2\dot{U} &= 2\rho_{oo} \theta \dot{S} + \frac{2}{1-\xi} \left\{ \frac{t_1 \dot{e}_1}{1-2e_1} + \frac{t_2 \dot{e}_2}{1-2e_2} \right\} \\ &= 2\rho_{oo} \theta \dot{S} + 2(U_1 + U_2 e_2) \dot{e}_1 + 2(U_1 + U_2 e_1) \dot{e}_2 \\ &= 2\rho_{oo} \theta \dot{S} + 2U_1 \frac{d}{d\eta}(e_1 + e_2) + 2U_2 \frac{d}{d\eta}(e_1 e_2) \\ &= 2\rho_{oo} \theta \dot{S} + 2U_1 \dot{e}_{11} - 4U_2 e_{12} \dot{e}_{12} , \end{aligned} \quad (4.5.36)$$

$$= 2\rho_{oo} \theta \dot{S} + 2U_1 (\dot{\xi} - \xi \dot{\xi} - \eta) - \eta U_2 , \quad (4.5.37)$$

where it was used that

$$\frac{d(e_1 + e_2)}{d\eta} = \dot{e}_{11} = \dot{\xi} - \xi \dot{\xi} - \eta ; \quad \frac{d}{d\eta}(e_1 e_2) = -2e_{12} \dot{e}_{12} = -\frac{1}{2}\eta , \quad (4.5.38)$$

which was obtained using Equations 4.5.25 and 4.5.8.

Equating the two values of $2\dot{U}$ from Equation 4.5.29 and 4.5.37 one obtains

$$\begin{aligned} 2\rho_{oo} \theta \dot{S} &= \frac{-d}{d\eta} \{ (\xi^2 + \eta^2) (U_1 + \frac{1}{2}U_2) - 2U_1 (\dot{\xi} - \xi \dot{\xi} - \eta) + U_2 \eta \\ &= -(\xi^2 + \eta^2) (\dot{U}_1 + \frac{1}{2}\dot{U}_2) - (2\dot{\xi}U_1 + \xi \dot{\xi}U_2) . \end{aligned} \quad (4.5.39)$$

Due to the requirement that ξ vanish with η , which is the single parameter defining the shock wave now, it is clear that the first invariant I also vanishes with η . So the condition that stress vanishes with strain demands that U_1 vanish with η . This is easily seen from Equation 4.5.32. Therefore it is clear that for $\eta=0$ Equation 4.5.39 yields

$$\dot{S}_0 = 0; \quad S = S(\eta); \quad (\rho_{00}, \theta_0 \neq 0), \quad (4.5.40)$$

where the suffix "o" means evaluation at $\eta=0$ and $S(0)=S_{00}$.

Differentiating Equation 4.5.39 with respect to η , one has

$$\begin{aligned} 2\rho_{00}(\dot{\theta}\dot{S}+\theta\ddot{S}) = & -(\xi^2+\eta^2)(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2) - (2\xi\dot{\xi}+2\eta)(\dot{U}_1 + \frac{1}{2}\dot{U}_2) \\ & - (2\ddot{\xi}\dot{U}_1 + \dot{\xi}^2 U_2 + \xi\ddot{\xi}U_2) - (2\dot{\xi}\dot{U}_1 + \xi\dot{\xi}\dot{U}_2). \end{aligned} \quad (4.5.41)$$

It is now recognized that since ξ is an even function of η and since ξ vanishes with η , then the assumed expansion for ξ given in Equation 4.5.18 gives that $\dot{\xi}$ vanishing with η ($\dot{\xi}_0 \equiv 0$). Further since U_1 and \dot{S} vanish with η , then Equation 4.5.41 evaluated at $\eta=0$ yields

$$\ddot{S}_0 = 0; \quad (\rho_{00}, \theta_0 \neq 0). \quad (4.5.42)$$

Another differentiation of Equation 4.5.41 with respect to η yields

$$\begin{aligned}
2\rho_{00}(\ddot{\theta}\dot{S}+2\dot{\theta}\ddot{S}+\ddot{\theta}\ddot{S})=-(\xi^2+\eta^2)(\ddot{U}_1+\frac{1}{2}\ddot{U}_2)-2(2\xi\dot{\xi}+\dot{\xi}+2\eta)\dot{U}_1 \\
-(3\xi\dot{\xi}+2\eta)\ddot{U}_2-2(\xi^2+\xi\ddot{\xi}+2\ddot{\xi}+1)\dot{U}_1-(3\dot{\xi}^2+3\xi\ddot{\xi}+1)\dot{U}_2 \\
-2\ddot{\xi}\dot{U}_1-(3\ddot{\xi}\ddot{\xi}+\xi\ddot{\xi})U_2.
\end{aligned} \tag{4.5.43}$$

The only nonvanishing terms are \dot{U}_1 and \dot{U}_2 ; however one has

$$\dot{U}_1=U_{11}\dot{I}+U_{12}\dot{I}\dot{I}+U_{1S}\dot{S}, \tag{4.5.44}$$

$$\dot{U}_2=U_{21}\dot{I}+U_{22}\dot{I}\dot{I}+U_{2S}\dot{S}. \tag{4.5.45}$$

From Equation 4.5.12, one can show \dot{I} and $I\dot{I}$ vanish with η , and since \dot{S} vanishes with η (Equation 4.5.40), then \dot{U}_1 and \dot{U}_2 also vanish with η . Thus evaluating Equation 4.5.43 at $\eta=0$ and further noting that \ddot{S} , ξ and $\dot{\xi}$ vanish with η , one obtains

$$\ddot{S}_0 = 0; \quad (\rho_{00}, \theta_0 \neq 0). \tag{4.5.46}$$

The final differentiation of Equation 4.5.43 with respect to η yields

$$\begin{aligned}
2\rho_{00}(\ddot{\theta}\dot{S}+3\dot{\theta}\ddot{S}+3\ddot{\theta}\ddot{S}+\ddot{\theta}\ddot{S})=-(\xi^2+\eta^2)(\ddot{U}_1+\frac{1}{2}\ddot{U}_2) \\
-2(3\xi\dot{\xi}+\dot{\xi}+3\eta)\ddot{U}_1-(4\xi\dot{\xi}+3\eta)\ddot{U}_2-6(\dot{\xi}^2+\xi\ddot{\xi}+\ddot{\xi}+1)\dot{U}_1 \\
-3(2\dot{\xi}^2+2\xi\ddot{\xi}+1)\ddot{U}_2-2(3\ddot{\xi}\ddot{\xi}+\xi\ddot{\xi}+3\ddot{\xi})\dot{U}_1
\end{aligned}$$

$$-4(\xi\ddot{\xi}+3\dot{\xi}\ddot{\xi})\dot{U}_2-2\ddot{\xi}\ddot{U}_1-(3\ddot{\xi}^2+4\dot{\xi}\ddot{\xi}+\xi\ddot{\xi})U_2. \quad (4.5.47)$$

Evaluate Equation 4.5.47 at $\eta=0$; since \dot{S} , \ddot{S} , $\ddot{\xi}$, $\dot{\xi}$, U_1 , \dot{U}_1 and \dot{U}_2 vanish with η , one obtains the final result that

$$2\rho_{\infty\theta\infty}\left(\frac{d^4S}{d\eta^4}\right)_0 = -3\{2(\ddot{\xi}+1)\ddot{U}_1+\ddot{U}_2+\ddot{\xi}^2U_2\}_0. \quad (4.5.48)$$

However the derivatives of U_1 and U_2 can be replaced by partial derivatives as follows: consider first \ddot{U}_1 ; it follows from Equation 4.5.44 that

$$\begin{aligned} \ddot{U}_1 &= \frac{d}{d\eta} (U_{11}\dot{I}+U_{12}\dot{I}\dot{I}+U_{1S}\dot{S}) \\ &= (U_{111}\dot{I}+U_{112}\dot{I}\dot{I}+U_{11S}\dot{S})\dot{I}+U_{11}\ddot{I}+(U_{121}\dot{I}+U_{122}\dot{I}\dot{I}+U_{12S}\dot{S})\dot{I}\dot{I} \\ &\quad + U_{12}\ddot{I}\dot{I}+(U_{1S1}\dot{I}+U_{1S2}\dot{I}\dot{I}+U_{1SS}\dot{S})\dot{S}+U_{1S}\ddot{S}. \end{aligned} \quad (4.5.49)$$

Noting that \dot{I} , $\dot{I}\dot{I}$, \dot{S} and \dot{S} vanish with η , Equation 4.5.49 evaluated at $\eta=0$ becomes

$$\ddot{U}_1)_0 = \{U_{11}\ddot{I}+U_{12}\ddot{I}\dot{I}\}_0. \quad (4.5.50)$$

From Equations 4.5.8 and 4.5.12, it follows that

$$\ddot{I})_0 = \ddot{\xi}_0 - 1; \quad \ddot{I}\dot{I})_0 = -\frac{1}{2}. \quad (4.5.51)$$

Using Equation 4.5.51 in Equation 4.5.50 yields

$$\ddot{U}_1)_0 = \{(\ddot{\xi}-1)U_{11} - \frac{1}{2}U_{12}\}_0 . \quad (4.5.52)$$

Similarly, $\ddot{U}_1)_0$ becomes

$$\ddot{U}_2)_0 = \{(\ddot{\xi}-1)U_{21} - \frac{1}{2}U_{22}\}_0 . \quad (4.5.53)$$

Combining Equations 4.5.52 and 4.5.53 with Equation 4.5.48 leads to

$$2\rho_{00}\theta_0 \left(\frac{d^4 S}{d\eta^4} \right)_0 = \{6U_{11} + 6U_{12} + \frac{3}{2}U_{22} - 3\ddot{\xi}^2(2U_{11} + U_2)\}_0 . \quad (4.5.54)$$

Equation 4.5.54 is valid whether U is a polynomial function or not. However in terms of the polynomial expansion for U given in Equation 4.4.53, Equation 4.5.54 can be written as

$$2\rho_{00}\theta_0 \left(\frac{d^4 S}{d\eta^4} \right)_0 = 3(p+2m+2\lambda+4\mu) - 6\ddot{\xi}_0^2(\lambda+\mu) . \quad (4.5.55)$$

Further using Equation 4.5.21 and 4.5.20, one obtains that

$$\ddot{\xi}_0 = \frac{m+2\lambda+8\mu}{2(\lambda+\mu)} . \quad (4.5.56)$$

Using Equation 4.5.56 in Equation 4.5.55, one obtains the final form as

$$2\rho_{00}\theta_0 \left(\frac{d^4 S}{d\eta^4} \right)_0 = 3\left\{p+2m+2\lambda+4\mu - \frac{(m+2\lambda+8\mu)^2}{2(\lambda+\mu)}\right\}_0 , \quad (4.5.57)$$

with all constants evaluated in the initial unstrained state.

Further using the expansion for entropy given in Equation 4.4.47, where ξ is replaced by η , and making use the same

assumptions given in the pure longitudinal case (e.g., η small but finite, etc.), one can now assert that entropy changes are of fourth-order in the strength of the shock, as measured here by η . This is equivalent to saying that entropy changes vary as the product of the square in density change from the first of Equation 4.5.5. This is a rather important but somewhat expected result. It shows that for $\eta = -\eta$, which is equivalent to applying a shear stress in the opposite direction, entropy changes are unaffected, which is an intuitively obvious result.

This study of a mixed transverse-longitudinal shock wave has revealed a number of features. Firstly there is no pure transverse shock wave. The only physically admissible shock in the case just considered is a mixed type with dominant transverse (shear) mode but accompanied by a weaker longitudinal (dilatational) mode. Secondly unless m is negative and very large (which for the case cited in Equation 4.4.37, it is not), then from Equation 4.5.22 and the density condition, k is positive and so the mixed transverse-longitudinal shock wave is always accompanied by expansion. Thirdly another important point that this study has brought out is the importance of p , called fourth-order elasticity. The nomenclature of ordering is misleading, indeed p is the coefficient of II^2 which is of the same order as I . Thus in this case of a predominantly transverse shock wave, the usual

nomenclature of ordering elasticities has little meaning. Further assuming Poisson's ratio, σ , as one-third and one-fourth gives the results

$$2\rho_{oo}\theta_o \left(\frac{d^4 S}{d\eta^4} \right)_o = 3\mu \left\{ \frac{p}{\mu} - 16 - 2\frac{m}{\mu} - \frac{1}{6} \left(\frac{m}{\mu} \right)^2 \right\}, \quad \sigma = \frac{1}{3}, \quad (4.5.58)$$

$$2\rho_{oo}\theta_o \left(\frac{d^4 S}{d\eta^4} \right)_o = 3\mu \left\{ \frac{p}{\mu} - 19 - 3\frac{m}{\mu} - \frac{1}{4} \left(\frac{m}{\mu} \right)^2 \right\}, \quad \sigma = \frac{1}{4}, \quad (4.5.59)$$

where $\sigma = \lambda/2(\lambda + \mu)$ was used.

Since entropy must increase the bracketed terms must be positive, indicating either m negative and p positive or if m is positive then p must be positive and quite large. In both cases p must be fairly large. Thus empirical determination must retain the term in p too. Most of these conclusions are of course dependent on the assumption of polynomial expansions; it may be more illuminating to consider other expansions which reduce to the first two terms in the linear theory.

In Equation 4.4.57 a known value of (m/μ) is cited; it is positive and small, which indicates in this case that (p/μ) must be fairly large.

4.6. General Three Dimensional Curved Shock Waves

It is the assertion of this section that no new types of shocks exist. Even in the case of curved three dimensional shocks moving in an initially unstrained medium at rest, one has only two types; the pure longitudinal shock wave and the mixed transverse-longitudinal shock wave, both being dilatational. The plane cases just considered expressed entropy changes only in terms of derivatives of the internal energy function U with respect to the strain invariants, evaluated in the initial state. Thus this depended only on the intrinsic properties of the medium itself. The truth of this statement for curved shocks implies that curvature of the shock-front does not enter the expression for entropy changes. A proof of the above assertion is given in the subsequent development and it is brought out there that the truth of this assertion can not be inferred from the tensorial character of the expressions alone. Indeed for the case of an initially strained medium, the result may be untrue and verification of the result in either way is by no means simple.

Let, as before, n_i be the unit normal to the wave-front pointing into the unstrained medium ahead and t_i and s_i be any two unit tangent vectors so that (n_i, t_i, s_i) form an orthonormal triad. The discontinuities in the displacements and the displacement gradients are given by (Equation

4.2.13)

$$[u_i] = 0; \quad [u_{i,j}] = \xi_i n_j; \quad \left[\frac{\partial u_i}{\partial t}\right] = -G\xi_i. \quad (4.6.1)$$

Consider now the shock conditions and the generalized Rankine-Hugoniot relation for an arbitrary curved three dimensional shock wave moving in an initially unstrained medium at rest in its natural state; these are given in Equations 4.2.32-4.2.36 as

$$[u_{i,j}] = u_{i,j}; \quad [v_i] = v_i; \quad [e_{ij}] = e_{ij}; \quad [t_{ij}] = t_{ij}, \quad (4.6.2)$$

from which one obtains

$$\rho_{oo} = \rho_o; \quad \rho = \rho_{oo}(1 - \xi_n); \quad t_{ji}n_j = \frac{\rho_{oo}G^2\xi_i}{1 - \xi_n}, \quad (4.6.3)$$

$$[U] = \frac{t_{ji}n_j\xi_i}{2(1 - \xi_n)}; \quad \xi_n = \xi_i n_i. \quad (4.6.4)$$

For plane shocks, $n_i = (1, 0, 0)$, which implies that the plane of the shock-front is parallel to the (x_2, x_3) plane, these reduce to

$$\rho = \rho_{oo}(1 - \xi_1); \quad \frac{\rho_{oo}G^2\xi_1}{1 - \xi_1} = t_{11}; \quad \frac{\rho_{oo}G^2\xi_2}{1 - \xi_1} = t_{12}, \quad (4.6.5)$$

$$\frac{\rho_{oo}G^2\xi_3}{1 - \xi_1} = t_{13}; \quad [U] = \frac{t_{11}\xi_1 + t_{12}\xi_2 + t_{13}\xi_3}{2(1 - \xi_1)}. \quad (4.6.6)$$

The assertion can now be stated that for curved shock waves, one has

$$\rho = \rho_{oo}(1 - \xi_n); \quad \frac{\rho_{oo} G^2 \xi_n}{1 - \xi_n} = t_{nn}; \quad \frac{\rho_{oo} G^2 \xi_t}{1 - \xi_n} = t_{nt}, \quad (4.6.7)$$

$$\frac{\rho_{oo} G^2 \xi_s}{1 - \xi_n} = t_{ns}; \quad [U] = \frac{t_{nn} \xi_n + t_{nt} \xi_t + t_{ns} \xi_s}{2(1 - \xi_n)} \quad (4.6.8)$$

where the subscripts n , t and s are clearly not tensor indices but denote normal and tangential components to the shock-front, as

$$\xi_n = \xi_i n_i; \quad \xi_t = \xi_i t_i; \quad e_{ij} n_i n_j = e_{nn},$$

$$e_{ij} n_i s_j = e_{ns}; \quad t_{ij} n_i n_j = t_{nn}; \quad t_{ij} n_i t_j = t_{nt}, \text{ etc.} \quad (4.6.9)$$

Further t_{nn} , t_{nt} and t_{ns} are shown to be the same functions of (ξ_n, ξ_t, ξ_s) as t_{11} , t_{12} and t_{13} are of (ξ_1, ξ_2, ξ_3) . It is therefore necessary to establish such a relationship in order to prove the assertion.

Consider first the strain tensor e_{ij} for an initially unstrained medium; one has

$$\begin{aligned} [e_{ij}] &= e_{ij} = \frac{1}{2}(\xi_i n_j + \xi_j n_i - \xi^2 n_i n_j) \\ &= (\xi_n - \frac{1}{2}\xi^2) n_i n_j + \frac{1}{2}\xi_t (n_i t_j + n_j t_i) + \frac{1}{2}\xi_s (n_i s_j + n_j s_i) \\ &= e_{nn} n_i n_j + e_{nt} (n_i t_j + n_j t_i) + e_{ns} (n_i s_j + n_j s_i), \end{aligned} \quad (4.6.10)$$

where it was used that $\xi_i = \xi_n n_i + \xi_t t_i + \xi_s s_i$ and $\xi^2 = \xi_n^2 + \xi_t^2 + \xi_s^2$.

A straight forward calculation now gives

$$\bar{\mathbf{e}} = \begin{pmatrix} e_{nn} & e_{nt} & e_{ns} \\ e_{nt} & 0 & 0 \\ e_{ns} & 0 & 0 \end{pmatrix}, \quad (4.6.11)$$

with

$$e_{nn} = \xi_n - \frac{1}{2}(\xi_n^2 + \xi_t^2 + \xi_s^2); \quad e_{nt} = \frac{1}{2}\xi_t; \quad e_{ns} = \frac{1}{2}\xi_s, \quad (4.6.12)$$

where the bars are temporarily used for tensors referred to the (n_i, t_i, s_i) system.

Using Equations 4.6.10-4.6.12, consider the strain invariants

$$I = e_{ii} = \frac{1}{2}\{\xi_i n_i + \xi_i n_i - \xi^2\} = (\xi_n - \frac{1}{2}\xi^2) = e_{nn} = \bar{I}, \quad (4.6.13)$$

$$\begin{aligned} II &= \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ji}) \\ &= \frac{1}{2}\{(\xi_n - \frac{1}{2}\xi^2)^2 - \frac{1}{4}(2\xi^2 + 2\xi_n^2 + \xi^4 - 4\xi^2\xi_n)\} \\ &= \frac{1}{4}\{\xi_n - \xi^2\} = -\frac{1}{4}(\xi_t^2 + \xi_s^2) \\ &= -(e_{nt}^2 + e_{ns}^2) = \bar{II}, \end{aligned} \quad (4.6.14)$$

$$III = \bar{III} = 0. \quad (4.6.15)$$

Further one needs the following result

$$\begin{aligned} e_{ik}e_{kj} &= \{(\xi_n - \frac{1}{2}\xi^2)n_i n_k + \frac{1}{2}\xi_t(n_i t_k + n_k t_i) + \frac{1}{2}\xi_s(n_i s_k + n_k s_i)\} \\ &\quad \cdot \{(\xi_n - \frac{1}{2}\xi^2)n_k n_j + \frac{1}{2}\xi_t(n_k t_j + n_j t_k) + \frac{1}{2}\xi_s(n_k s_j + n_j s_k)\} \end{aligned}$$

$$\begin{aligned}
&= \{(\xi_n - \frac{1}{2}\xi^2) + \frac{1}{2}\xi_t + \frac{1}{2}\xi_s\}n_i n_j + \frac{1}{2}\xi_t(\xi_n - \frac{1}{2}\xi^2)(n_i t_j + n_j t_i) \\
&\quad + \frac{1}{2}\xi_s(\xi_n - \frac{1}{2}\xi^2)(n_i s_j + n_j s_i) + \frac{1}{4}\xi_t^2 t_i t_j + \frac{1}{4}\xi_s^2 s_i s_j \\
&\quad + \frac{1}{4}\xi_s \xi_t (t_i s_j + t_j s_i) \\
&= (\bar{e}^2)_{nn} n_i n_j + (\bar{e}^2)_{nt} (n_i t_j + n_j t_i) + (\bar{e}^2)_{ns} (n_i s_j + n_j s_i) \\
&\quad + (\bar{e}^2)_{tt} t_i t_j + (\bar{e}^2)_{ss} s_i s_j + (\bar{e}^2)_{st} (t_i s_j + t_j s_i),
\end{aligned}
\tag{4.6.16}$$

where the subscripts above denote the corresponding positions in the matrix product (\bar{e}^2) ; e.g., $(\bar{e}^2)_{nn}$ is the first row, first column entry of (\bar{e}^2) , etc. Thus \bar{e}^2 is the same function of (ξ_n, ξ_t, ξ_s) as e^2 is of (ξ_1, ξ_2, ξ_3) .

Then obtaining \bar{t}_{ij} from t_{ij} by use of the tensor transformation rule as

$$\begin{aligned}
t_{ij} &= t_{nn} n_i n_j + t_{nt} (n_i t_j + n_j t_i) + t_{ns} (n_i s_j + n_j s_i) \\
&\quad + t_{st} (s_i t_j + s_j t_i) + t_{ss} s_i s_j + t_{tt} t_i t_j.
\end{aligned}
\tag{4.6.17}$$

which upon substitution into the following expression for t_{ij} as a function of e_{ij} ,

$$t = a\delta + b\epsilon + c\epsilon^2, \tag{4.6.18}$$

one arrives at

$$\bar{t} = a\bar{\xi} + b\bar{\xi} + c\bar{\xi}^2, \quad (4.6.19)$$

where Equations 4.6.10 and 4.6.16 were used.

Thus \bar{t} is the same function of $\bar{\xi}$ as t is of ξ . So even in the three dimensional case, the formal shock conditions, the generalized Rankine-Hugoniot relation and Gibb's equation remain the same functions of (ξ_n, ξ_t, ξ_s) as these were of (ξ_1, ξ_2, ξ_3) in the plane case. Thus the assertion is proved. Therefore all the conclusions obtained in the plane case continue to be valid for any arbitrary three dimensional curved shock.

To recognize the need of this proof, consider the simple expansion for the gradient of the displacement vector, given by

$$\begin{aligned} u_{i,j} &= \{n_j \frac{\partial}{\partial n} + t_j \frac{\partial}{\partial t} + s_j \frac{\partial}{\partial s}\} \{un_i + vt_i + ws_i\} \\ &= \left\{ \frac{\partial u}{\partial n} n_i n_j + \frac{\partial v}{\partial n} n_j t_i + \frac{\partial w}{\partial n} n_j s_i \right\} \\ &\quad + \{n_j (u \frac{\partial n_i}{\partial n} + v \frac{\partial t_i}{\partial n} + w \frac{\partial s_i}{\partial n}) \\ &\quad + (t_j \frac{\partial}{\partial t} + s_j \frac{\partial}{\partial s}) (un_i + vt_i + ws_i)\}. \end{aligned} \quad (4.6.20)$$

The symbols follow the previously defined nomenclature of this section. Now it must be noted that it is only the terms in the first curly bracket in Equation 4.6.20 that can suffer

jumps in an initially unstrained medium. The terms in the second curly bracket are continuous due to the continuity of u_i , Hadamard's lemma and since the medium is initially unstrained. So that when the jump in $u_{i,j}$ is taken only the first group of terms in the first curly bracket survive. But whatever form of strain tensor used, in all the non-linear forms, it is not only the gradients that appear but also the product of the gradients that contribute to the strain. The jump in the product is contributed not only by the product of the jumps but also by the quantities ahead, as

$$[PQ] = [P][Q] - P_0[Q] - Q_0[P] . \quad (4.6.21)$$

Thus the strains will involve the group of terms in the second curly bracket in Equation 4.6.20 in the strained ahead case. However these involve not only initial strains but also curvature effects; e.g. $[\partial n_i / \partial n]$, etc. It is only in the case of an initially unstrained medium that this group always drops out. So any shock wave travelling into a medium which is initially unstrained does not depend on the curvature of the shock; however this is no longer true for an arbitrary three dimensional curved shock wave travelling into an initially strained medium.

This assertion only relates to the shock conditions, the generalized Rankine-Hugoniot relation and Gibb's equation and so for the calculation of entropy changes.

As was mentioned earlier a large group of workers [3-5, 8-12] adopt the gas-dynamical model "fluid model" for a solid in the study of elastic shock waves. The only explanation given seems to be that for strong shocks, shear is unimportant and hence the gas-dynamical model is reasonable. Duvall [3] has discussed at great length how to generate plane shocks and also points out that the equipment necessary to produce such shocks must be very sophisticated and is very costly. The measurements that are usually taken from these experiments are shock velocity and free surface velocity of the specimen, from which they can determine the material's equation of state assuming a "fluid model".

One of the objects of this study was to determine if such a fluid model follows from an exact nonlinear elastic theory. The results of this study thus far show that the curved pure longitudinal shock, which is defined by $\xi_1 = (\xi_n, 0, 0)$, is formally exactly similar to the gas-dynamical model; though pressure is not isotropic, it does not affect the problem at all at the shock-front; which justifies the gas-dynamical model used by this group of workers [3-5, 8-12].

4.7. Shock Waves in an Incompressible Hyperelastic Medium

Plane shocks moving in an incompressible hyperelastic medium which is initially unstrained and at rest in its natural state are considered first. Then it is shown that the results obtained in the plane case are also valid for curved three dimensional shocks. The completely spatial formulation that is used here has revealed some interesting results which raises questions on commonly used forms of the strain energy function.

As before for plane shocks, choose $n_i = (1, 0, 0)$ and $\xi_i = (\xi, \eta, \zeta)$. Further since the medium is initially unstrained and at rest one has

$$\rho_{00} = \rho_0; \quad [u_{i,j}] = u_{i,j}; \quad [v_i] = v_i, \quad (4.7.1)$$

$$[e_{ij}] = e_{ij}; \quad [t_{ij}] = t_{ij}. \quad (4.7.2)$$

Further the discontinuities in the displacement gradients, given in Equation 4.2.13, again are

$$[u_i] = 0; \quad [u_{i,j}] = \xi_i n_j; \quad \left[\frac{\partial u_i}{\partial t} \right] = -G \xi_i. \quad (4.7.3)$$

Again a certain rotation of the 2-3 axes always exists to secure ξ_i in the form $(\xi, \eta, 0)$ in the new coordinate system. So without loss of generality the shock conditions, given in Equations 4.4.2 and 4.4.3, reduce to

$$\rho = \rho_{00}(1-\xi); \quad \frac{\rho_{00}G^2\xi}{1-\xi} = t_{11}; \quad \frac{\rho_{00}G^2\eta}{1-\xi} = t_{12} . \quad (4.7.4)$$

However since the material is incompressible, $\rho = \rho_{00} = \rho_0$ and thus from the first of Equation 4.7.4 $\xi \equiv 0$. This implies $t_{11} = 0$ which gives an equation for determining the pressure P from the constitutive equation. Thus from Equation 4.7.4 it follows that the only possible shock wave is the pure transverse shock wave, which is also pure shear. Therefore in an incompressible medium there can exist only one type of shock, viz. a pure transverse shock wave. For this wave longitudinal amplitude vanishes though longitudinal strain does not.

A clarification with regard to the terminology may be in order. It is conventional in gas-dynamics to name a shock as one for which $[\rho(v_n - G)] \neq 0$. Here a shock wave is taken to mean, among other things, as a wave across which at least one of the displacement gradients is discontinuous. Indeed as seen below, it does have an amplitude dependent normal speed of propagation.

From Equation 4.7.3, the equations of motion now yield only one equation,

$$\rho_{00}G^2\eta = t_{12}; \quad \xi_i = (0, \eta, 0) . \quad (4.7.5)$$

From Equation 4.4.4, the generalized Rankine-Hugoniot relation reduces to

$$[U] = \frac{t_{12}\eta}{2} . \quad (4.7.6)$$

From Equations 4.4.5 and 4.4.6, the strain tensor and its components reduce to

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e^2 = \begin{pmatrix} e_{11}^2 + e_{12}^2 & e_{11}e_{12} & 0 \\ e_{11}e_{12} & e_{12}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.7.7)$$

with

$$e_{11} = -\frac{1}{2}\eta^2; \quad e_{12} = \frac{1}{2}\eta, \quad (4.7.8)$$

and from Equation 4.4.13, the invariants become

$$I = e_{11}; \quad II = -e_{12}^2; \quad III = 0. \quad (4.7.9)$$

From Equations 4.4.7, 3.5.46 and 3.5.47, the stress tensor and its components reduce to

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (4.7.10)$$

with

$$t_{11} = -P + be_{11} + c(e_{11}^2 + e_{12}^2); \quad t_{12} = -e_{12}(2U_1 + U_2), \quad (4.7.11)$$

$$t_{22} = -P + ce_{12}^2 ; \quad t_{33} = -P , \quad (4.7.12)$$

where $U_1 = U_1(I, II, S)$, $U_2 = U_2(I, II, S)$ and the last of Equation 4.7.11 is obtained from Equation 4.4.9. Also the material function a is assumed absorbed in P .

Use of the second of Equation 4.7.11 in the generalized Rankine-Hugoniot relation in Equation 4.7.6, yields

$$[U] = - \frac{(U_1 + \frac{1}{2} U_2) \eta^2}{2} , \quad (4.7.13)$$

or

$$\dot{U} = -(U_1 + \frac{1}{2} U_2) \eta - \frac{1}{2} (\dot{U}_1 + \frac{1}{2} \dot{U}_2) \eta^2 , \quad (4.7.14)$$

where the dot represents the total derivative with respect to the strength of the shock, as measured here by η .

The principal strains, which are needed in Gibb's equation, are given in Equation 4.5.25; they are:

$$e_1 = \frac{1}{2} \{ e_{11} + (e_{11}^2 + 4e_{12}^2)^{1/2} \} ; \quad e_2 = \frac{1}{2} \{ e_{11} - (e_{11}^2 + 4e_{12}^2)^{1/2} \} ,$$

$$e_3 = 0 , \quad (4.7.15)$$

which for this case are functions of η only.

The principal stresses given in Equation 3.5.45, become

$$t_1 = -P + (1 - 2e_1) (U_1 + e_2 U_2) , \quad (4.7.16)$$

$$t_2 = -P + (1 - 2e_2) (U_1 + e_1 U_2) . \quad (4.7.17)$$

Combining Equations 4.7.16, 4.7.17 and 3.5.44, Gibb's

equation reduces to

$$\begin{aligned}\dot{U} &= \rho_{oo} \theta \dot{S} + (U_1 + e_2 U_2) \dot{e}_1 + (U_1 + e_1 U_2) \dot{e}_2 \\ &= \rho_{oo} \theta \dot{S} + U_1 \frac{d}{d\eta} (e_1 + e_2) + U_2 \frac{d}{d\eta} (e_1 e_2),\end{aligned}\quad (4.7.18)$$

which, after using Equations 4.7.15 and 4.5.38, becomes

$$\begin{aligned}\dot{U} &= \rho_{oo} \theta \dot{S} + U_1 \dot{e}_{11} - 2U_2 e_{12} \dot{e}_{12} \\ &= \rho_{oo} \theta \dot{S} - (U_1 + \frac{1}{2} U_2) \eta.\end{aligned}\quad (4.7.19)$$

Equating values of \dot{U} in Equations 4.7.14 and 4.7.19, one has

$$2\rho_{oo} \theta \dot{S} = -(\dot{U}_1 + \frac{1}{2} \dot{U}_2) \eta^2. \quad (4.7.20)$$

From Equation 4.7.20 it is obvious that \dot{S} vanishes with η , thus

$$\dot{S}_o = 0; \quad S = S(\eta); \quad (\rho_{oo}, \theta_o \neq 0), \quad (4.7.21)$$

where the suffix "o" means evaluation at $\eta=0$ and $S(0)=S_{oo}$.

Differentiate Equation 4.7.20 with respect to η to obtain

$$2\rho_{oo} (\theta \dot{S} + \dot{\theta} S) = -2(\dot{U}_1 + \frac{1}{2} \dot{U}_2) \eta - (\ddot{U}_1 + \frac{1}{2} \ddot{U}_2) \eta^2. \quad (4.7.22)$$

Again it is obvious that \ddot{S} vanishes with η , thus

$$\ddot{S}_O = 0; \quad (\rho_{OO}, \theta_O \neq 0) . \quad (4.7.23)$$

Differentiating Equation 4.7.22 with respect to η again yields

$$\begin{aligned} 2\rho_{OO}(\ddot{\theta}\dot{S} + 2\dot{\theta}\ddot{S} + \ddot{\theta}\ddot{S}) = & -(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2)\eta^2 - 4(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2)\eta \\ & - 2(\dot{U}_1 + \frac{1}{2}\dot{U}_2) . \end{aligned} \quad (4.7.24)$$

The only nonvanishing terms, possibly, are \dot{U}_1 and \dot{U}_2 ; however one has

$$\dot{U}_1 = U_{11}\dot{I} + U_{12}\dot{I}\dot{I} + U_{1S}\dot{S} , \quad (4.7.25)$$

$$\dot{U}_2 = U_{21}\dot{I} + U_{22}\dot{I}\dot{I} + U_{2S}\dot{S} . \quad (4.7.26)$$

From Equation 4.7.8, it is easily shown that

$$\dot{I}_O = 0; \quad \dot{I}\dot{I}_O = 0 . \quad (4.7.27)$$

Further, from Equation 4.7.21, $\dot{S}_O = 0$, therefore it follows that

$$\dot{U}_1)_O = 0; \quad \dot{U}_2)_O = 0. \quad (4.7.28)$$

Thus evaluating Equation 4.7.24 at $\eta=0$ and noting that \dot{S} , \ddot{S} , \dot{U}_1 and \dot{U}_2 vanish with η , gives

$$\ddot{S}_O = 0 ; \quad (\rho_{OO}, \theta_O \neq 0) . \quad (4.7.29)$$

A final differentiation of Equation 4.7.24 with respect

to η yields

$$2\rho_{oo}(\ddot{\theta}\dot{S}+3\ddot{\theta}\ddot{S}+3\dot{\theta}\ddot{S}+\dot{\theta}\ddot{S}) = -(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2)\eta^2 \\ -6(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2)\eta - 6(\ddot{U}_1 + \frac{1}{2}\ddot{U}_2) . \quad (4.7.30)$$

Evaluating Equation 4.7.30 at $\eta=0$ and noting that \dot{S} , \ddot{S} and \ddot{S} vanish with η , yields

$$2\rho_{oo}\theta_o \left(\frac{d^4 S}{d\eta^4} \right)_o = -3(2\ddot{U}_1 + \ddot{U}_2)_o . \quad (4.7.31)$$

Equation 4.7.31 can be expressed in terms of a derivative of t_{12} . If one differentiates the second of Equation 4.7.11 three times with respect to η , one obtains

$$\ddot{t}_{12} = -\eta \frac{(2\ddot{U}_1 + \ddot{U}_2)}{2} - \frac{3}{2}(2\ddot{U}_1 + \ddot{U}_2) \quad (4.7.32)$$

Evaluating Equation 4.7.32 at $\eta=0$ yields

$$2\ddot{t}_{12})_o = -3(2\ddot{U}_1 + \ddot{U}_2)_o . \quad (4.7.33)$$

Using Equation 4.7.33 in Equations 4.7.31, one obtains

$$\rho_{oo}\theta_o \left(\frac{d^4 S}{d\eta^4} \right)_o = \frac{\partial^3 t_{12}}{\partial \eta^3} \Big|_o . \quad (4.7.34)$$

As before the total derivatives of U_1 and U_2 in Equation 4.7.31 can be replaced by partial derivatives. Differentiating Equations 4.7.24 and 4.7.25 with respect to η and evaluating at $\eta=0$ yields

$$\ddot{U}_1)_0 = \{U_{11}\ddot{I} + U_{12}\ddot{I}\ddot{I}\}_0, \quad (4.7.35)$$

$$\ddot{U}_2)_0 = \{U_{12}\ddot{I} + U_{22}\ddot{I}\ddot{I}\}_0. \quad (4.7.36)$$

Using Equations 4.7.8 and 4.7.9, it is easily verified that

$$\ddot{I})_0 = -1; \quad \ddot{I}\ddot{I})_0 = -\frac{1}{2}. \quad (4.7.37)$$

Combining Equations 4.7.35, 4.7.36 and 4.7.37 with Equation 4.7.31, one obtains the final result

$$2\rho_{00}\theta_0 \left(\frac{d^4 S}{d\eta^4} \right)_0 = 3(2U_{11} + 2U_{12} + \frac{1}{2}U_{22})_0. \quad (4.7.38)$$

Now using the expansion for entropy given in Equation 4.4.47, where ξ is replaced by η , and making use of the same assumptions as before, one can assert that the entropy changes are fourth-order in the shock-strength, as measured here by η . Further one imposes the physical requirement that the entropy must increase. This implies, from Equations 4.4.47 and 4.7.38, that

$$3(2U_{11} + 2U_{12} + \frac{1}{2}U_{22})_0 \eta^4 > 0, \quad (4.7.39)$$

or

$$3(2U_{11} + 2U_{12} + \frac{1}{2}U_{22})_0 > 0. \quad (4.7.40)$$

The one commonly accepted energy function for generalized Mooney materials ([59], p. 213] reduces, in terms of the present strain tensor, to the form (see Appendix B)

$$U = 4\alpha(II-I) + f(-2I) . \quad (4.7.41)$$

The condition given in Equation 4.7.40 implies

$$\left. \frac{d^2 f}{dI^2} \right|_0 > 0. \quad (4.7.42)$$

It is usual to take f as a constant times its argument (or zero in the case of a neo-hookean material) and then entropy is conserved across a shock! Based perhaps on experimental results, this arbitrary function, f , is taken to be a slowly varying function of its argument; so, in order to obtain analytical solutions, it is taken as a constant (small) times its argument [68]. Written in terms of the present notation, this argument involves only the first invariant. However for a linear theory the first invariant vanishes for incompressibility; thus the slowly varying nature of the function, f , for moderately small but finite strains appears to be reasonable!

Further it is clear, from the proof given in section 4.6 for an arbitrary curved three dimensional shock wave travelling in an initially unstrained compressible medium at rest, that the results of the incompressible shock wave just considered are also valid for an arbitrary curved three dimensional shock wave travelling in an initially unstrained incompressible medium at rest.

5. ENTROPY INEQUALITIES: ISOTROPIC MEDIUM-INITIALLY STRAINED AND AT REST

5.1. General Considerations

The full generality of the results obtained in the previous chapter for an initially unstrained medium were not obtained in the present case of an initially strained medium. The general problem in this case has not proved tractable. In the initially unstrained case, classification of shocks as pure longitudinal and mixed type was achieved for the general problem of an arbitrary three-dimensional curved shock wave, as well as for the plane shock wave. The corresponding general problem is an arbitrarily initially strained medium has not thus far yielded to such classification. Leaving such a question for further study, only the "principal shock waves" are studied here. These are defined analogous to "principal waves" [66].

The most general situation considered here is the case of plane shock waves moving in an arbitrary but uniformly strained medium at rest. Since the initial strain is constant at all points, the principal directions do not vary from point to point. Thus to simplify mathematical difficulties, one can choose the reference spatial rectangular coordinate axes to coincide with the principal axes of the initial strain. Hence one can choose the same fixed system at all

points. This introduces a lot of simplification into the analysis. With this choice, the initial displacement gradients and initial strains, given in Equation 4.2.10, are

$$u_{oi,j} = \begin{pmatrix} u_{o1,1} & 0 & 0 \\ 0 & u_{o2,2} & 0 \\ 0 & 0 & u_{o3,3} \end{pmatrix}; \quad e_{oij} = \begin{pmatrix} e_{o11} & 0 & 0 \\ 0 & e_{o22} & 0 \\ 0 & 0 & e_{o33} \end{pmatrix}. \quad (5.1.1)$$

In this chapter, one zero as a suffix denotes the initially strained medium (medium ahead) and two zeros as a suffix denote the material's natural unstrained state. All the necessary equations for this chapter have been derived in section 4.2 of this work and will be referred to where necessary.

If ρ_o , ρ_{oo} are the densities in the initially strained and natural states respectively, from Equation 4.2.21, one has

$$\begin{aligned} \rho_o &= \rho_{oo} \{ (1-2e_{o11}) (1-2e_{o22}) (1-2e_{o33}) \}^{1/2} \\ &= \rho_{oo} (1-u_{o1,1}) (1-u_{o2,2}) (1-u_{o3,3}) \\ &= \frac{\rho_{oo}}{\lambda_{o1} \lambda_{o2} \lambda_{o3}}, \end{aligned} \quad (5.1.2)$$

with

$$(1-u_{o1,1}) = \frac{1}{\lambda_{o1}} ; (1-u_{o2,2}) = \frac{1}{\lambda_{o2}} ; (1-u_{o3,3}) = \frac{1}{\lambda_{o3}} , \quad (5.1.3)$$

where λ_{o1} , λ_{o2} and λ_{o3} are the principal stretches for the strained ahead medium defined in Equation 4.4.33.

As before let the discontinuity vector ξ_i in the gradient of the displacement vector be given by (Equation 4.2.13)

$$[u_i] = 0; [u_{i,j}] = \xi_i n_j; \left[\frac{\partial u_i}{\partial t}\right] = -G\xi_i, \quad (5.1.4)$$

where

$$\xi_i = (\xi_1, \xi_2, \xi_3). \quad (5.1.5)$$

Then the shock conditions, which are obtained from the continuity equation and the balance of linear momentum, the consistency condition and the generalized Rankine-Hugoniot relation, given in Equations 4.2.17, 4.2.19, 4.2.26, 4.2.27, 4.2.28, 4.2.29 and 4.2.31, are

$$\rho = \rho_o(1-\bar{\xi}_n) ; v_i = \frac{-G\bar{\xi}_i}{1-\bar{\xi}_n} ; \bar{\xi}_n = \bar{\xi}_i n_i , \quad (5.1.6)$$

$$[f_i] = [t_{ji}]n_j = (t_{ji}-t_{oji})n_j = \frac{\rho_o G^2 \bar{\xi}_i}{1-\bar{\xi}_n} , \quad (5.1.7)$$

$$\frac{[f_1]}{\bar{\xi}_1} = \frac{[f_2]}{\bar{\xi}_2} = \frac{[f_3]}{\bar{\xi}_3} ; [U] = \frac{\rho_{oo} \bar{\xi}_i (f_i + f_{oi})}{2\rho_o(1-\bar{\xi}_n)} , \quad (5.1.8)$$

where

$$\bar{\xi}_1 = \frac{\xi_1}{1-u_{01,1}} ; \bar{\xi}_2 = \frac{\xi_2}{1-u_{02,2}} ; \bar{\xi}_3 = \frac{\xi_3}{1-u_{03,3}} . \quad (5.1.9)$$

For plane shocks, choose again $n_i=(1,0,0)$ and $\xi_i=(\xi,\eta,\zeta)$. Further, since the reference spatial coordinate axes is chosen to coincide with the principal axes of the initial strain, which defines principal shocks, one cannot rotate, as before, to secure ξ_i in the form $(\xi,\eta,0)$. Therefore principal shocks with the discontinuity vector $\xi_i=(\xi,\eta,0)$ are only discussed here.

For the case considered it is now possible to assert that there exists only two types of principal shocks. The first is the pure longitudinal type given by $\xi_i=(\xi,0,0)$ and the second is the mixed type given by $\xi_i=(\xi,\eta,0)$. The proof of this assertion follows directly from the proof given for plane shocks in an unstrained medium at rest. Further there are in all nine principal shocks in number but they are only of these two types.

The strain tensor, given in Equation 4.2.24, is now given by

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix} , \quad (5.1.10)$$

with

$$e_{11} = e_{011} + (\bar{\xi} - \frac{1}{2} \bar{\xi}^2) (1-u_{01,1})^2 - \frac{1}{2} \bar{\eta}^2 (1-u_{02,2})^2 , \quad (5.1.11)$$

$$e_{12} = \frac{1}{2} \bar{\eta} (1-u_{02,2})^2 ; \quad e_{22} = e_{022} ; \quad e_{33} = e_{033} , \quad (5.1.12)$$

where

$$\bar{\xi} = \frac{\xi}{1-u_{01,1}} ; \quad \bar{\eta} = \frac{\eta}{1-u_{02,2}} . \quad (5.1.13)$$

The strain invariants, given in Equations 3.3.3 and 3.3.4 and 3.3.5, in this case reduce to

$$I = I_0 + e_{11} - e_{011} , \quad (5.1.14)$$

$$II = II_0 + (e_{022} + e_{033})(e_{11} - e_{011}) - e_{12}^2 , \quad (5.1.15)$$

$$III = III_0 + e_{022}e_{033}(e_{11} - e_{011}) - e_{033}e_{12}^2 . \quad (5.1.16)$$

The stress tensor and its components are obtained from Equation 3.4.22 using Equations 5.1.14, 5.1.15 and 5.1.16.

They are:

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix} , \quad (5.1.17)$$

with

$$\begin{aligned} t_{11} = & \frac{\rho}{\rho_{00}} \{ U_1 (1-\bar{\xi})(1-u_{01,1})^2 + \bar{\eta}^2 (1-u_{02,2})^2 \} \\ & + U_2 \{ (e_{022} + e_{033}) \{ (1-\bar{\xi})^2 (1-u_{01,1})^2 \\ & + \bar{\eta}^2 (1-u_{02,2})^2 \} + \frac{1}{2} \bar{\eta}^2 (1-u_{02,2})^4 \} \end{aligned}$$

$$\begin{aligned}
& + U_3 \left\{ \frac{1}{2} e_{033} \bar{\eta}^2 (1 - u_{02,2})^4 + e_{022} e_{033} \{ (1 - \bar{\xi})^2 (1 - u_{01,1})^2 \right. \\
& \left. + \bar{\eta}^2 (1 - u_{01,1})^2 \} \right\}, \tag{5.1.18}
\end{aligned}$$

$$t_{12} = - \frac{\rho}{\rho_{00}} \bar{\eta} (1 - u_{02,2})^2 \left\{ U_1 + \frac{1}{2} U_2 (1 + 2e_{033}) + \frac{1}{2} U_3 e_{033} \right\}, \tag{5.1.19}$$

$$\begin{aligned}
t_{22} = \frac{\rho}{\rho_{00}} \{ & U_1 (1 - 2e_{022}) + U_2 \{ (I - e_{022}) (1 - 2e_{022}) + 2e_{12}^2 \} \\
& + U_3 \{ II - 2III - e_{022} (I - e_{022}) + e_{12}^2 \} \}, \tag{5.1.20}
\end{aligned}$$

$$\begin{aligned}
t_{33} = \frac{\rho}{\rho_{00}} \{ & U_1 (1 - 2e_{033}) + U_2 (I - e_{033}) (1 - 2e_{033}) \\
& + U_3 \{ II - 2III - e_{033} (I - e_{033}) \} \}, \tag{5.1.21}
\end{aligned}$$

where

$$U_1 = U_1(I, II, III, S) = \left. \frac{\partial U}{\partial I} \right|_{II, III, S = \text{constant}}, \text{ etc.} \tag{5.1.22}$$

5.2. Plane Principal Shock Waves of the Pure Longitudinal Type

From Equations 5.1.7, 5.1.8, 5.1.19, 5.1.20 and 5.1.21, it can be seen it is admissible to take $\eta = \xi = \bar{\eta}$ which gives $\xi_i = (\xi, 0, 0)$. This reduces Equation 5.1.19 and the second of Equation 5.1.7 to identities. Further Equations 5.1.6, 5.1.7 and 5.1.8 now reduce to

$$\rho = \rho_o (1 - \bar{\xi}) ; \frac{\rho_o G^2 \bar{\xi}}{1 - \bar{\xi}} = [t_{11}] = (t_{11} - t_{o11}), \quad (5.2.1)$$

$$[U] = \frac{\rho_{oo} \bar{\xi} (t_{11} + t_{o11})}{2 \rho_o (1 - \bar{\xi})}, \quad (5.2.2)$$

where

$$\bar{\xi} = \frac{\xi}{1 - u_{o1,1}}. \quad (5.2.3)$$

The strain tensor and its components, given in Equations 5.1.10, 5.1.11 and 5.1.12, reduce to

$$e_{ij} = \begin{pmatrix} e_{11} & 0 & 0 \\ 0 & e_{o22} & \\ 0 & 0 & e_{o33} \end{pmatrix}, \quad (5.2.4)$$

with

$$e_{11} = e_{o11} + \left(\bar{\xi} - \frac{1}{2} \bar{\xi}^2 \right) (1 - u_{o1,1})^2. \quad (5.2.5)$$

The strain invariants from Equations 5.1.14, 5.1.15 and 5.1.16 reduce to

$$I = I_o + \left(\bar{\xi} - \frac{1}{2} \bar{\xi}^2 \right) (1 - u_{o1,1})^2, \quad (5.2.6)$$

$$II = II_o + (e_{o22} + e_{o33}) (e_{11} - e_{o11}), \quad (5.2.7)$$

$$III = III_o + e_{o22} e_{o33} (e_{11} - e_{o11}). \quad (5.2.8)$$

The stress tensor and its components, given in Equations 5.1.17, 5.1.18, 5.1.19, 5.1.20 and 5.1.21, reduce to

$$t_{ij} = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (5.2.9)$$

with

$$\begin{aligned} t_{11} = & \frac{\rho}{\rho_{00}} \{U_1 (1-\bar{\xi}) (1-u_{01,1})^2 \\ & + U_2 (e_{022} + e_{033}) (1-\bar{\xi})^2 (1-u_{01,1})^2 \\ & + U_3 e_{022} e_{033} (1-\bar{\xi})^2 (1-u_{01,1})^2\}, \end{aligned} \quad (5.2.10)$$

$$\begin{aligned} t_{22} = & \frac{\rho}{\rho_{00}} \{U_1 (1-2e_{022}) + U_2 (I - e_{022}) (1-2e_{022}) \\ & + U_3 \{II - 2III - e_{022} (I - e_{022})\}\}, \end{aligned} \quad (5.2.11)$$

$$\begin{aligned} t_{33} = & \frac{\rho}{\rho_{00}} \{U_1 (1-2e_{033}) + U_2 (I - e_{033}) (1-2e_{033}) \\ & + U_3 \{II - 2III - e_{033} (I - e_{033})\}\}. \end{aligned} \quad (5.2.12)$$

From Equations 5.2.4 and 5.2.9, an obvious result is

$$e_1 = e_{11}; \quad e_2 = e_{022}; \quad e_3 = e_{033}, \quad (5.2.13)$$

$$t_1 = t_{11}; \quad t_2 = t_{22}; \quad t_3 = t_{33}, \quad (5.2.14)$$

where e_a and t_a are the principal strains and stresses respectively.

Gibb's equation, given in Equation 3.5.30 reduces to

$$\begin{aligned} dU &= \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho_o (1-\bar{\xi})} \left\{ \frac{t_{11} de_{11}}{1-2e_{11}} \right\} , \\ &= \rho_{oo} \theta dS + \frac{\rho_{oo} t_{11} d\xi}{\rho_o (1-\bar{\xi})^2 (1-u_{o1,1})} , \end{aligned} \quad (5.2.15)$$

or

$$\dot{U} = \rho_{oo} \theta \dot{S} + \frac{\rho_{oo} t_{11}}{\rho_o (1-\bar{\xi})^2 (1-u_{o1,1})} , \quad (5.2.16)$$

where the dot again represents the total derivatives with respect to the shock-strength, as measured here by ξ (not $\bar{\xi}$).

Equating values of \dot{U} from Equations 5.2.2 and 5.6.16, one obtains after some simplification

$$2\rho_o \theta \dot{S} = \frac{\bar{\xi} \dot{t}_{11}}{1-\bar{\xi}} + \frac{t_{o11} - t_{11}}{(1-\bar{\xi})^2 (1-u_{o1,1})} . \quad (5.2.17)$$

Evaluating at $\xi=0$ and noting that $t_{11})_o = t_{o11}$, it is clear that

$$\dot{S}_o = 0, \quad (\rho_o, \theta_o \neq 0), \quad (5.2.18)$$

where the suffix "o" means evaluation at $\xi=0$ and $S(0)=S_o$.

Differentiate Equation 5.2.17 with respect to ξ and simplifying one has

$$2\rho_o(\dot{\theta}\dot{S}+\theta\ddot{S}) = \frac{\bar{\xi} \ddot{t}_{11}}{1-\bar{\xi}} + \frac{2(t_{o11}-t_{11})}{(1-\bar{\xi})^2(1-u_{o1,1})^2} . \quad (5.2.19)$$

Evaluating at $\xi=0$, it is again obvious that

$$\ddot{S}_o=0, \quad (\rho_o, \theta_o \neq 0) . \quad (5.2.20)$$

A final differentiation with respect to ξ yields

$$\begin{aligned} 2\rho_o(\ddot{\theta}\dot{S}+2\dot{\theta}\ddot{S}+\theta\ddot{\ddot{S}}) &= \frac{\bar{\xi} \ddot{t}_{11}}{1-\bar{\xi}} + \frac{\ddot{t}_{11}}{(1-\bar{\xi})^2(1-u_{o1,1})} \\ &\quad - \frac{2\dot{t}_{11}}{(1-\bar{\xi})^3(1-u_{o1,1})^2} + \frac{6(t_{o11}-t_{11})}{(1-\bar{\xi})^4(1-u_{o1,1})^3} . \end{aligned} \quad (5.2.21)$$

Evaluating at $\xi=0$, gives the final result

$$2\rho_o\theta_o \left(\frac{d^3S}{d\xi^3} \right)_{\xi=0, S=S_o} = \left(\frac{\ddot{t}_{11}}{1-u_{o1,1}} - \frac{2t_{11}}{(1-u_{o1,1})^2} \right)_{\xi=0, S=S_o} . \quad (5.2.22)$$

Since $e_{11}=e_{o11} + (\bar{\xi} - \frac{1}{2}\bar{\xi}^2)(1-u_{o1,1})^2$, Equation 5.2.22 can be written as

$$2\rho_o\theta_o \left(\frac{d^3S}{d\xi^3} \right)_o = \left(\frac{1}{\lambda_1} \frac{\partial^2 t_{11}}{\partial e_{11}^2} - 3\lambda_1 \frac{\partial t_{11}}{\partial e_{11}} \right)_o , \quad (5.2.23)$$

where the λ_1 is the principal stretch in the one direction, which is given in Equation 4.4.34.

One can also write Equation 5.2.22 in terms of derivatives of the internal energy function, as was done for the

initially unstrained case. However, it is not very illuminating. However it assumes a very simple form when written in terms of the tension-stretch law. To this end, using Equation 4.4.34, consider the results

$$(1-2e_1) = (1-2e_{11}) = (1-u_{1,1})^2 = \frac{1}{\lambda_1^2}, \quad (5.2.24)$$

$$\frac{d\lambda_1}{d\xi} = \lambda_1^{-2}; \quad \frac{d^2\lambda_1}{d\xi^2} = 2\lambda_1^{-3}. \quad (5.2.25)$$

From these results and the first of Equation 5.1.3, one has

$$\begin{aligned} \dot{t}_{11}|_0 &= \lambda_{01}^{-2} \left. \frac{\partial t_{11}}{\partial \lambda_1} \right|_0; \quad \ddot{t}_{11}|_0 = \lambda_{01}^{-4} \left. \frac{\partial^2 t_{11}}{\partial \lambda_1^2} \right|_0 \\ &\quad + 2\lambda_{01}^{-3} \left. \frac{\partial t_{11}}{\partial \lambda_1} \right|_0. \end{aligned} \quad (5.2.26)$$

Use of Equation 5.2.26 in Equation 5.2.22, one obtains the result

$$2\rho_0 \theta_0 \left. \frac{d^3 S}{d\xi^3} \right|_{\substack{\xi=0 \\ S=S_0}} = \lambda_{01}^{-5} \left. \frac{\partial^2 t_{11}}{\partial \lambda_1^2} \right|_{\substack{\xi=0 \\ S=S_0}}, \quad (5.2.27)$$

which reduces to Equation 4.4.35, which however is more general since it is true for curved shock waves too.

Now using the expansion for entropy given in Equation 4.4.47 and making use the same assumptions as before, one can assert that entropy changes are of third-order in the shock-

strength, as measured here by ξ ; which is exactly the same result that was obtained in the initially unstrained case. Further one imposes the requirement that entropy must increase which yields

$$\lambda_{01}^5 \frac{\partial^2 t_{11}}{\partial \lambda_1^2} \bigg|_0 > 0 . \quad (5.2.28)$$

It is interesting to note that Truesdell [66] in his study of acceleration waves shows that the slope of the tension-stretch curve must be positive if the speed of propagation is to be real (his result was obtained using the undeformed reference system; it can be proved using the present configuration too). The above result shows that the curvature of this curve must be positive.

The results obtained here do not depend on the polynomial expansion of the internal energy function and hence have a wider range of validity. Indeed, for a strained medium, such an expansion consists of an infinite series, even when valid.

5.3. Plane Principal Shock Waves of the Mixed Transverse-Longitudinal Type

In section 5.1 it was shown that there exists only two types of plane principal shocks; the pure longitudinal one, which was just considered, and the mixed transverse-longitudinal type which is to be discussed here. For this type, the discontinuity in the displacement gradients is taken

to be of the form

$$\xi_i = (\xi, \eta, 0); \quad n_i = (1, 0, 0) . \quad (5.3.1)$$

As was mentioned earlier, for the case considered here, there are only two types of plane principal shock waves, the pure longitudinal type, which yields three shock waves and the above mentioned mixed type which yields six shock waves, giving nine in number. Some of these are:

$$\begin{aligned} \xi_i &= (\xi, \eta, 0); \quad n_i = (0, 1, 0) , \\ \xi_i &= (0, \eta, \zeta) \quad n_i = (0, 0, 1) , \\ \xi_i &= (0, 0, \zeta); \quad n_i = (0, 0, 1); \text{ etc.} \end{aligned}$$

From Equations 5.1.6, 5.1.7 and 5.1.8, the shock conditions, the consistency condition and the generalized Rankine-Hugoniot relation become for this case

$$\rho = \rho_o (1 - \bar{\xi}); \quad v_i = \frac{-G \bar{\xi}_i}{1 - \bar{\xi}_n}; \quad (t_{11} - t_{o11}) = \frac{\rho_o G^2 \bar{\xi}}{1 - \bar{\xi}}, \quad (5.3.2)$$

$$t_{12} = \frac{\rho_o G^2 \bar{\eta}}{1 - \bar{\xi}}; \quad \frac{t_{11} - t_{o11}}{\bar{\xi}} = \frac{t_{12}}{\bar{\eta}}, \quad (5.3.3)$$

$$[\bar{U}] = \frac{\rho_{oo} \{ \bar{\xi} (t_{11} + t_{o11}) + \bar{\eta} t_{12} \}}{2 \rho_o (1 - \bar{\xi})}, \quad (5.3.4)$$

where

$$\bar{\xi} = \frac{\xi}{1 - u_{o1,1}}; \quad \bar{\eta} = \frac{\eta}{1 - u_{o2,2}} . \quad (5.3.5)$$

The strain tensor and its components given in Equations 5.1.10, 5.1.11 and 5.1.12 reduce to

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{022} & 0 \\ 0 & 0 & e_{033} \end{pmatrix}, \quad (5.3.6)$$

$$e_{11} = e_{011} + (\bar{\xi} - \frac{1}{2} \bar{\xi}^2) (1 - u_{01,1})^2 - \frac{1}{2} \bar{\eta}^2 (1 - u_{02,2})^2, \quad (5.3.7)$$

$$e_{12} = \frac{1}{2} \bar{\eta} (1 - u_{02,2})^2. \quad (5.3.8)$$

The strain invariants are given in Equations 5.1.14, 5.1.15, and 5.1.16. They are:

$$I = I_0 + e_{11} - e_{011}, \quad (5.3.9)$$

$$II = II_0 + (e_{022} + e_{033})(e_{11} - e_{011}) - e_{12}^2, \quad (5.3.10)$$

$$III = III_0 + e_{022}e_{033}(e_{11} - e_{011}) - e_{033}e_{12}^2. \quad (5.3.11)$$

The principal strains will also be needed; they are given by

$$2e_1 = e_{11} + e_{022} + \{(e_{11} - e_{022})^2 + 4e_{12}^2\}^{1/2}, \quad (5.3.12)$$

$$2e_2 = e_{11} + e_{022} - \{(e_{11} - e_{022})^2 + 4e_{12}^2\}^{1/2}, \quad (5.3.13)$$

$$e_3 = e_{033}. \quad (5.3.14)$$

The stress tensor and its components are given in Equations 5.1.17, 5.1.18 and 5.1.19. They are:

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (5.3.15)$$

with

$$\begin{aligned} t_{11} = & \frac{\rho}{\rho_{00}} \{U_1 \{(1-\bar{\xi})^2 (1-u_{01,1})^2 + \bar{\eta}^2 (1-u_{02,2})^2\} \\ & + U_2 \{(e_{022} + e_{033}) \{(1-\bar{\xi})^2 (1-u_{01,1})^2 \\ & + \bar{\eta}^2 (1-u_{02,2})^2\} + \frac{1}{2} \bar{\eta}^2 (1-u_{02,2})^4\} \\ & + U_3 \{\frac{1}{2} e_{033} \bar{\eta}^2 (1-u_{02,2})^4 + e_{022} e_{033} \{(1-\bar{\xi})^2 (1-u_{01,1})^2 \\ & + \bar{\eta}^2 (1-u_{02,2})^2\}, \end{aligned} \quad (5.3.16)$$

$$t_{12} = - \frac{\rho}{\rho_{00}} \bar{\eta} (1-u_{02,2})^2 \{U_1 + \frac{1}{2} U_2 (1+2e_{033}) + \frac{1}{2} U_3 e_{033}\}, \quad (5.3.17)$$

$$t_{011} = \frac{\rho_0}{\rho_{00}} (1-u_{01,1})^2 \{U_1 + U_2 (e_{022} + e_{033}) + U_3 e_{022} e_{033}\}_0, \quad (5.3.18)$$

and t_{22} and t_{33} are given in Equations 5.1.20 and 5.1.21.

The principal stresses will also be needed. Combining Equation 3.5.28 with the strain invariants written in terms of the principal strains, one obtains

$$t_1 = \frac{\rho}{\rho_{00}} (1-2e_1) \{U_1 + U_2 (e_2 + e_{033}) + U_3 e_2 e_{033}\}, \quad (5.3.19)$$

$$t_2 = \frac{\rho}{\rho_{00}} (1-2e_2) \{U_1 + U_2(e_1 + e_{033}) + U_3 e_1 e_{033}\}, \quad (5.3.20)$$

$$t_3 = t_{033}, \quad (5.3.21)$$

where

$$U_1 = U_1(I, II, III, S) = \left. \frac{\partial U}{\partial I} \right|_{II, III, S = \text{constant}}, \quad \text{etc.} \quad (5.3.22)$$

First note that, from Equations 5.3.16 and 5.3.17, $t_{11} = t_{11}(\xi, \eta, S)$ and $t_{12} = t_{12}(\xi, \eta, S)$. It is therefore clear that one cannot take $t_{11}(0, \eta, S) = 0$, which is in contrast to the pure longitudinal case where it was possible to take $\xi \neq 0$, $\eta = 0$. Therefore a pure shear shock wave with $\eta \neq 0$ and $\xi = 0$ is not possible in the initially strained case also. Further it is again asserted that ξ is an even function of η . This is easily seen by writing the consistency condition in terms of derivatives of the internal energy function. The consistency condition, which reduces a given shock to a one parameter family, is obtained by equating the two normal speeds G given by the third of Equation 5.3.2 and the first of Equation 5.3.3. If one substitutes the expressions for t_{11} , t_{011} and t_{12} in terms of the derivatives of the internal energy function given in Equations 5.3.16, 5.3.17 and 5.3.18 into the consistency condition given in the second of Equation 5.3.3, one obtains

$$\begin{aligned}
& \frac{1}{\bar{\xi}} \{U_1 \{ (1-\bar{\xi})^2 (1-u_{01,1})^2 + \bar{\eta}^2 (1-u_{02,2})^2 \} \\
& + U_2 \{ (e_{022} + e_{033}) \{ (1-\bar{\xi})^2 (1-u_{01,1})^2 \\
& + \bar{\eta}^2 (1-u_{02,2})^2 \} + \frac{1}{2} \bar{\eta}^2 (1-u_{02,2})^4 \} \\
& + U_3 \{ e_{022} e_{033} \{ (1-\bar{\xi})^2 (1-u_{01,1})^2 + \bar{\eta}^2 (1-u_{02,2})^2 \} \\
& + \frac{1}{2} e_{033} \bar{\eta}^2 (1-u_{02,2})^4 \} - \frac{(1-u_{01,1})^2}{\bar{\xi}(1-\bar{\xi})} \{ U_1 + U_2 (e_{022} + e_{033}) \\
& + U_3 e_{022} e_{033} \} \\
& = -(1-u_{02,2})^2 \{ U_1 + \frac{1}{2} U_2 (1+2e_{033}) + \frac{1}{2} U_3 e_{033} \}. \quad (5.3.23)
\end{aligned}$$

Equation 5.3.23 gives that ξ must be an even function of η in order that the consistency equation be satisfied. This then completes the proof of the initial assertion that there are only two types of principal shock waves; viz. the pure longitudinal shock wave and the mixed type shock wave. It is further noted that both are dilatational.

The case of $\xi = o(\eta)$ as $\eta \rightarrow 0$ is only considered here. This is the case of a dominant transverse mode accompanied by a weaker longitudinal mode and is called a transverse-longitudinal principal shock wave; the first adjective, which is here transverse, is always used in this study to indicate the stronger mode.

The case of $\eta = o(\xi)$ as $\xi \rightarrow 0$, which was argued as unphysical

in the initially unstrained case, has not yielded a similar such result in the initially strained case. Using arguments similar to those used in Equation 5.5.20, in Equation 5.3.23, has not yielded thus far results from which its admissibility or non-admissibility can be argued. This is an area for future study. Here again insufficient knowledge about the form of the internal energy function has been the major difficulty.

Thus as before ξ is assumed to be of the form given in Equation 4.5.18. Further for η small but finite ($\eta < 1$), it is assumed that

$$\xi = k(s)\eta^2 + O(\eta^4) + \dots, \quad (5.3.24)$$

or

$$\bar{\xi} = k_o(s)\bar{\eta}^2 + O(\bar{\eta}^4) + \dots, \quad (5.3.25)$$

where

$$k_o(s) = k(s) \frac{(1-u_{o2,2})^2}{(1-u_{o1,1})^2}. \quad (5.3.26)$$

The generalized Rankine-Hugoniot relation given in Equation 5.3.4 is now combined with the consistency condition to obtain

$$[U] = \rho_{oo} \frac{(\bar{\xi}^2 + \bar{\eta}^2)t_{12}}{2\rho_o \bar{\eta}(1-\bar{\xi})} + \frac{\rho_{oo} t_{o11} \bar{\xi}}{\rho_o (1-\bar{\xi})}. \quad (5.3.27)$$

To study entropy changes across the shock it is useful to write the Rankine-Hugoniot relation in terms of the derivatives of the internal energy function. Use of Equation

5.3.17 with Equation 5.3.27 yields

$$[U] = - \frac{(\bar{\xi}^2 + \bar{\eta}^2) (1 - u_{o2,2})^2}{2} \{U_1 + \frac{1}{2}U_2(1+2e_{o33}) + \frac{1}{2}U_3e_{o33}\} \\ + \frac{\rho_{oo} t_{o11} \dot{\bar{\xi}}}{\rho_o (1 - \bar{\xi})} . \quad (5.3.28)$$

or

$$2\dot{U} = -(\bar{\xi}^2 + \bar{\eta}^2) (1 - u_{o2,2})^2 \{\dot{U}_1 + \frac{1}{2}\dot{U}_2(1+2e_{o33}) + \frac{1}{2}\dot{U}_3e_{o33}\} \\ - \{2\bar{\xi} \dot{\bar{\xi}} (1 - u_{o2,2})^2 + 2\bar{\eta}\} \{U_1 + \frac{1}{2}U_2(1+2e_{o33}) + \frac{1}{2}U_3e_{o33}\} \\ + \frac{2\rho_{oo} t_{o11} \dot{\bar{\xi}}}{\rho_o (1 - \bar{\xi})} + \frac{2\rho_{oo} t_{o11} \bar{\xi} \dot{\bar{\xi}}}{\rho_o (1 - \bar{\xi})^2} , \quad (5.3.29)$$

where the dot represents the total derivative with respect to the strength of the shock, as measured here by η .

One finally needs Gibb's equation to study entropy changes. From Equation 3.5.30, it reduces for this case to

$$dU = \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho} \left\{ \frac{t_1 de_1}{1-2e_1} + \frac{t_2 de_2}{1-2e_2} \right\}. \quad (5.3.30)$$

Using the expressions for t_1 and t_2 given in Equations 5.3.19 and 5.3.20, Gibb's equation can be shown to reduce to

$$\dot{U} = \rho_{oo} \theta \dot{S} + \{U_1 + U_2 e_{o33}\} \frac{de_{11}}{d\eta} + \{U_2 + U_3 e_{o33}\} \frac{d(e_{11}e_{o22} - e_{12}^2)}{d\eta},$$

or

$$\dot{U} = \rho_{oo} \theta \dot{S} + (U_1 + U_2 e_{o33}) \{(\dot{\bar{\xi}} - \bar{\xi} \dot{\bar{\xi}}) (1 - u_{o1,1})^2 - \bar{\eta}\} \\ + (U_2 + U_3 e_{o33}) \{(\dot{\bar{\xi}} - \bar{\xi} \dot{\bar{\xi}}) (1 - u_{o1,1})^2 - \frac{1}{2}\bar{\eta}\}, \quad (5.3.31)$$

where it was used that

$$\frac{d}{d\eta}(e_1+e_2) = \dot{e}_{11} = (\bar{\xi}-\bar{\xi} \frac{\dot{\xi}}{\xi})(1-u_{01,1})^{2-\eta},$$

$$\frac{d}{d\eta}(e_1 e_2) = \dot{e}_{11} e_{022} - 2e_{12} \dot{e}_{12} = (\bar{\xi}-\bar{\xi} \frac{\dot{\xi}}{\xi})(1-u_{01,1})^2 e_{022} - \frac{1}{2}\eta,$$

which was obtained from Equations 5.3.7, 5.3.8, 5.3.12 and 5.3.13.

Equating the two values of \dot{U} from Equations 5.3.29 and 5.3.31 yields

$$\begin{aligned} 2\rho_{oo}\theta\dot{S} = & -(\bar{\xi}^2+\bar{\eta}^2)(1-u_{02,2})^2\{\dot{U}_1 + \frac{1}{2}\dot{U}_2(1+2e_{033}) + \frac{1}{2}\dot{U}_3e_{033}\} \\ & -\{2\bar{\xi}\frac{\dot{\xi}}{\xi}(1-u_{02,2})^{2+\eta}\}\{U_1 + \frac{1}{2}U_2(1+2e_{033}) + \frac{1}{2}U_3e_{033}\} \\ & -2(U_1+U_2e_{033})\{(\bar{\xi}-\bar{\xi}\frac{\dot{\xi}}{\xi})(1-u_{01,1})^{2-\eta}\} \\ & -2(U_2+U_3e_{033})\{(\bar{\xi}-\bar{\xi}\frac{\dot{\xi}}{\xi})(1-u_{01,1})^2e_{022} - \frac{1}{2}\eta\} \\ & + \frac{2\rho_{oo}t_{011}\frac{\dot{\xi}}{\xi}}{\rho_o(1-\bar{\xi})} + \frac{2\rho_{oo}t_{011}\bar{\xi}\frac{\dot{\xi}}{\xi}}{\rho_o(1-\bar{\xi})^2}. \end{aligned} \quad (5.3.32)$$

Due to the requirement that ξ vanish with η and since ξ is even in η , then $\bar{\xi}$ and $\frac{\dot{\xi}}{\xi}$ vanish with η . Evaluating Equation 5.3.32 at $\eta=0$ yields

$$\dot{S}_o = 0, \quad (\rho_{oo}, \theta_o \neq 0), \quad (5.3.33)$$

where the suffix "o" means evaluation at $\eta=0$ and $S(0)=S_o$.

The differentiations are hereon lengthy but straight-

forward. Differentiating twice with respect to η and evaluating at $\eta=0$ yields

$$\ddot{S}_0 = 0, \quad (\rho_{00}, \theta_0 \neq 0), \quad (5.3.34)$$

$$2\rho_{00}\theta_0\ddot{S}_0 = -\{2\dot{U}_1 + \dot{U}_2(1+2e_{033}) + \dot{U}_3e_{033}\}_0 \\ -4\ddot{\xi}_0\{\dot{U}_1 + \dot{U}_2(e_{022}+e_{033}) + \dot{U}_3e_{022}e_{033}\}_0. \quad (5.3.35)$$

However one has

$$\dot{U}_1 = U_{11}\dot{I} + U_{12}\dot{II} + U_{13}\dot{III} + U_{1S}\dot{S}, \quad (5.3.36)$$

$$\dot{U}_2 = U_{21}\dot{I} + U_{22}\dot{II} + U_{23}\dot{III} + U_{2S}\dot{S}, \quad (5.3.37)$$

and from Equations 5.3.9-5.3.11 it can be shown that

$\dot{I}_0 = \dot{II}_0 = \dot{III}_0 = 0$. Therefore it is concluded that

$\dot{U}_1)_0 = \dot{U}_2)_0 = \dot{U}_3)_0 = 0$. Hence Equation 5.3.35 reduces to

$$\ddot{S}_0 = 0, \quad (\rho_{00}, \theta_0 \neq 0). \quad (5.3.38)$$

A final differentiation with respect to η and evaluation at $\eta=0$ and noting that $\ddot{\xi}_0 = \ddot{\xi}_0 = 0$ and $\dot{S}_0 = \ddot{S}_0 = \ddot{S}_0 = 0$, one obtains

$$2\rho_{00}\theta_0\left(\frac{d^4S}{d\eta^4}\right)_0 = -3\{2\ddot{U}_1 + \ddot{U}_2(1+2e_{033}) + \ddot{U}_3e_{033}\}_0 \\ -6\ddot{\xi}_0(1-u_{01,1})\{\ddot{U}_1 + \ddot{U}_2(e_{022}+e_{033}) + \ddot{U}_3e_{022}e_{033}\}_0 \\ -18\ddot{\xi}_0^2\{U_1 + U_2(e_{022}+e_{033}) + U_3e_{022}e_{033}\}_0 \\ -3\ddot{\xi}_0^2\frac{(1-2e_{022})}{(1-2e_{011})}\{2U_1 + U_2(1+2e_{033}) + U_3e_{033}\}_0, \quad (5.3.39)$$

which reduces to Equation 4.5.43, if $u_{01,1} = u_{02,2} = u_{03,3} = 0$.

Now the derivatives of U_1 , U_2 and U_3 with respect to η can be replaced by partial derivatives with respect to the strain invariants. If one uses the same procedure used in the initially unstrained case, Equation 5.3.39 can be shown to simplify to

$$\begin{aligned}
 2\rho_{00}\theta_0 \left. \frac{d^4 S}{d\eta^4} \right|_{\substack{\xi=0 \\ S=S_0}} &= -6\{U_{11}\{\ddot{\xi}_0^2(1-2e_{011})-1\} \\
 &+ U_{12}\{2\ddot{\xi}_0^2(1-2e_{011})(e_{022}+e_{033})-(1+2e_{033})\} \\
 &+ U_{22}\{\ddot{\xi}_0^2(1-2e_{011})(e_{022}+e_{033})^2\} \\
 &- \frac{1}{4}(1+2e_{033})^2\} \\
 &+ U_{13}e_{033}\{2\ddot{\xi}_0^2e_{022}(1-2e_{011})-1\} \\
 &+ U_{23}e_{033}\{4\ddot{\xi}_0^2e_{022}(1-2e_{011})(e_{022}+e_{033}) \\
 &- (1+2e_{033})\} \\
 &+ U_{33}e_{033}^2\{\ddot{\xi}_0^2e_{022}^2(1-2e_{011})-\frac{1}{4}\} \\
 &- 3\ddot{\xi}_0^2\{U_1+U_2(e_{022}+e_{033})+U_3e_{022}e_{033}\} \\
 &+ \frac{\ddot{\xi}_0^2(1-2e_{022})}{2(1-2e_{011})}\{2U_1+U_2(1+2e_{033})+U_3e_{033}\}\}_{0'}
 \end{aligned}
 \tag{5.3.40}$$

which also reduces to the initially unstrained result.

This result is very general in the sense that no expansions have been assumed for U . Further using the expansion for entropy given in Equation 4.4.47, where ξ is replaced by η , and making use of the same assumptions that were used there, one can now assert that entropy changes are of fourth-order in the strength of the shock wave, as measured here by η . It is further noted that this is the same result that was obtained for the initially unstrained case.

One further imposes the physical requirement that entropy must increase across the shock; which gives that the right hand side of Equation 5.3.40 must be positive.

As in the initially unstrained case, a Taylor series expansion can be assumed for the internal energy function U . However, if it is assumed, then it must be expanded about the initial strain state and not about the zero (unstrained) state.

5.4. Plane Principal Shock Waves in an Incompressible Hyperelastic Medium

Since the initial strain is constant at all points, the reference spatial coordinate axes in the shocked material is again chosen to coincide with the principal axes of the initial strain. Thus one can again discuss principal shocks.

For plane shocks in general, one can choose $n_i = (1, 0, 0)$ and $\xi_i = (\xi_1, \xi_2, \xi_3)$. However, the case considered here is when $\xi_i = (\xi_1, \xi_2, 0)$ or $\xi_i = (\xi_1, 0, \xi_3)$. The case with $\xi_i = (\xi_1, \xi_2, \xi_3)$ gives another pure transverse shock but is not discussed here.

The shock conditions given in Equations 5.1.6 and 5.1.7 now take the form

$$\rho = \rho_0(1 - \bar{\xi}_1); \quad \frac{\rho_0 G^2 \bar{\xi}_1}{1 - \bar{\xi}_1} = t_{11} - t_{011}; \quad \frac{\rho_0 G^2 \bar{\xi}_2}{1 - \bar{\xi}_1} = t_{12}, \quad (5.4.1)$$

where

$$\bar{\xi}_1 = \frac{\xi_1}{1 - u_{01,1}}; \quad \bar{\xi}_2 = \frac{\xi_2}{1 - u_{02,2}}. \quad (5.4.2)$$

However incompressibility demands that $\rho = \rho_0 = \rho_{00}$ which, from the first of Equation 5.4.1, leads to $\bar{\xi}_1 = 0$. This implies $t_{11} - t_{011} = 0$ which gives an equation determining the pressure P . Thus the only possible shock wave is the pure transverse shock wave, which is also a pure shear shock wave. Therefore in an incompressible medium there can exist only one type of shock wave.

From Equations 5.4.1, the equations of motion reduce to only one equation

$$\rho_0 G^2 \bar{\eta} = t_{12}; \quad \xi_i = (0, \eta, 0), \quad (5.4.3)$$

where

$$\bar{\eta} = \frac{\eta}{1 - u_{02,2}} \quad (5.4.4)$$

From the second of Equation 5.1.8, the Rankine-Hugoniot relation becomes

$$[U] = \frac{\bar{\eta} t_{12}}{2}, \quad (5.4.5)$$

where now $U=U(I,II,S)$. As mentioned earlier there are only two independent strain invariants taken here as I and II .

The strain tensor and its components, given in Equations 5.1.10-12, become

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{o22} & 0 \\ 0 & 0 & e_{o33} \end{pmatrix}, \quad (5.4.6)$$

with

$$e_{11} = e_{o11} - \frac{1}{2} \eta^2; \quad e_{12} = \frac{1}{2} \eta (1 - u_{o2,2}). \quad (5.4.7)$$

The strain invariants, given in Equations 5.1.14-16, become

$$I = I_o + (e_{11} - e_{o11}), \quad (5.4.8)$$

$$II = II_o + (e_{o22} + e_{o33})(e_{11} - e_{o11}) - e_{12}^2 \quad (5.4.9)$$

The stress tensor, given in Equation 5.1.17, and its components, which are obtained from Equation 3.5.46 and 3.5.47 become

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (5.4.10)$$

with

$$t_{11} = -P + be_{11} + c(e_{11}^2 + e_{12}^2), \quad (5.4.11)$$

$$t_{12} = -\eta(1-u_{o2,2})\{U_1 + \frac{1}{2}U_2(1+2e_{o33})\}, \quad (5.4.12)$$

$$t_{22} = -P + be_{o22} + c(e_{12}^2 + e_{o22}^2), \quad (5.4.13)$$

$$t_{33} = t_{o33} = -P + be_{o33} + ce_{o33}^2. \quad (5.4.14)$$

The principal strains, which will be needed later, are given in Equations 5.3.12-14. They are:

$$2e_1 = (e_{11} + e_{o22}) + \{(e_{11} - e_{o22})^2 + 4e_{12}^2\}^{1/2}, \quad (5.4.15)$$

$$2e_2 = (e_{11} + e_{o22}) - \{(e_{11} - e_{o22})^2 + 4e_{12}^2\}^{1/2}, \quad (5.4.16)$$

$$e_3 = e_{o33}. \quad (5.4.17)$$

From Equation 3.5.45, the principal stresses in terms of derivatives of $U=U(I,II,S)$ reduce to

$$t_1 = -P + (1-2e_1)\{U_1 + U_2(e_2 + e_{o33})\}, \quad (5.4.18)$$

$$t_2 = -P + (1-2e_2)\{U_1 + U_2(e_1 + e_{o33})\}. \quad (5.4.19)$$

Gibb's equation, given in Equation 3.5.44, reduces to

$$dU = \rho_{oo} \theta ds + \frac{(t_1+P)}{1-2e_1} de_1 + \frac{(t_2-P)}{1-2e_2} de_2 , \quad (5.4.20)$$

or

$$\dot{U} = \rho_{oo} \theta \dot{S} + \frac{(t_1+P)}{1-2e_1} \frac{de_1}{d\eta} + \frac{(t_2+P)}{1-2e_2} \frac{de_2}{d\eta} , \quad (5.4.21)$$

where the dot again represents the total derivative with respect to the strength of the shock, as measured here by η .

Using Equations 5.4.18 and 5.4.19 in Equation 5.4.21 yields

$$\dot{U} = \rho_{oo} \theta \dot{S} + (U_1+U_2 e_{o33}) \frac{d}{d\eta}(e_1+e_2) + U_2 \frac{d}{d\eta}(e_1 e_2) . \quad (5.4.22)$$

Further, using Equations 5.4.15 and 5.4.16 with Equation 5.4.7, Equations 5.4.22 can be reduced to

$$\dot{U} = \rho_{oo} \theta \dot{S} - \eta \left\{ U_1 + \frac{1}{2} U_2 (1+2e_{o33}) \right\} . \quad (5.4.23)$$

Substituting the expression for t_{12} from Equation 5.4.12 into Equation 5.4.5 and then differentiating with respect to η , one obtains

$$\dot{U} = -\frac{1}{2} \eta^2 \left\{ \dot{U}_1 + \frac{1}{2} \dot{U}_2 (1+2e_{133}) \right\} - \eta \left\{ U_1 + \frac{1}{2} U_2 (1+2e_{o33}) \right\} . \quad (5.4.24)$$

Equating values of \dot{U} from Equations 5.4.22 and 5.4.24 and simplifying, one obtains

$$2\rho_{oo}\dot{\theta}\dot{S} = -\eta^2\{\dot{U}_1 + \frac{1}{2}\dot{U}_2(1+2e_{o33})\} . \quad (5.4.25)$$

Evaluating Equation 5.4.23 at $\eta=0$ yields

$$\dot{S}_o = 0, \quad (\rho_{oo}, \theta_o \neq 0) , \quad (5.4.26)$$

where the suffix "o" means evaluation at $\eta=0$ and $S(0)=S_{oo}$.

It is obvious from the form of Equation 5.4.23 that \dot{S} vanishes with η , since \dot{S} is of order η^2 . Then a third derivative of Equation 5.4.23 yields

$$2\rho_{oo}(\ddot{\theta}\dot{S} + 2\dot{\theta}\ddot{S} + \ddot{\theta}\dot{S}) = -\eta^2\{\ddot{U}_1 + \frac{1}{2}\ddot{U}_2(1+2e_{o33})\} \\ \eta\{\ddot{U}_1 + \frac{1}{2}\ddot{U}_2(1+2e_{o33})\} - 2\{\dot{U}_1 + \frac{1}{2}\dot{U}_2(1+2e_{o33})\} . \quad (5.4.27)$$

The only possible nonvanishing term at $\eta=0$ is the term involving \dot{U}_1 and \dot{U}_2 . However one has

$$\dot{U}_1 = U_{11}\dot{I} + U_{12}\dot{I}\dot{I} + U_{1S}\dot{S} , \quad (5.4.28)$$

$$\dot{U}_2 = U_{21}\dot{I} + U_{22}\dot{I}\dot{I} + U_{2S}\dot{S} , \quad (5.4.29)$$

and from Equations 5.4.7, 5.4.8 and 5.4.9, it is clear that $\dot{\mathbf{I}}_0 = \dot{\mathbf{II}}_0 = 0$, which implies $\dot{\mathbf{U}}_1)_0 = \dot{\mathbf{U}}_2)_0 = 0$. Therefore Equation 5.4.27

$$\ddot{\mathbf{S}}_0 = 0, \quad (\rho_{00}, \theta_0 \neq 0). \quad (5.4.30)$$

A final differentiation and evaluation at $\eta=0$ of Equation 5.4.27 yields

$$2\rho_{00}\theta_0 \left. \frac{d^4 \mathbf{S}}{d\eta^4} \right|_0 = -6\{\ddot{\mathbf{U}}_1 + \frac{1}{2}\ddot{\mathbf{U}}_2(1+2e_{033})\}_0. \quad (5.4.31)$$

The total derivatives of \mathbf{U}_1 and \mathbf{U}_2 with respect to η can be replaced by partial derivatives with respect to the strain invariants as follows: Differentiate Equations 5.4.28 and 5.4.29 with respect to η and then evaluate at $\eta=0$. If one further makes use of Equations 5.4.7, 5.4.8 and 5.4.9 and noting that ξ , $\dot{\xi}$, $\dot{\mathbf{S}}$ and $\ddot{\mathbf{S}}$ vanish with η , one obtains

$$\ddot{\mathbf{U}}_1)_0 = (\mathbf{U}_{11}\ddot{\mathbf{I}} + \mathbf{U}_{12}\ddot{\mathbf{II}})_0, \quad (5.4.32)$$

$$\ddot{\mathbf{U}}_2)_0 = (\mathbf{U}_{21}\ddot{\mathbf{I}} + \mathbf{U}_{22}\ddot{\mathbf{II}})_0, \quad (5.4.33)$$

with

$$\ddot{\mathbf{I}}_0 = -1; \quad \ddot{\mathbf{II}}_0 = -\frac{1}{2}(1+2e_{033}). \quad (5.4.34)$$

Combining these results with Equation 5.4.31, one finally obtains

$$2\rho_{00}\theta_0 \left. \frac{d^4 S}{d\eta^4} \right|_{\substack{\xi=0 \\ S=S_0}} = 3\{2U_{11}+2U_{12}(1+2e_{033}) + \frac{1}{2}U_{22}(1+2e_{033})^2\}_0, \quad (5.4.35)$$

which exactly reduces to Equation 4.7.38.

A further simplification of Equation 5.4.31 yields a very interesting simple form. If one differentiates the expression for t_{12} , given in Equation 5.4.12, three times with respect to η and evaluates at $\eta=0$, one obtains

$$\ddot{t}_{12})_0 = -3(1-u_{02,2})\{\ddot{U}_1 + \frac{1}{2}\ddot{U}_2(1+2e_{033})\}_0. \quad (5.4.36)$$

Combining Equation 5.4.36 with Equation 5.4.31 and using the principal stretches which are defined in Equation 4.4.34, one obtains

$$2\rho_{00}\theta_0 \left. \frac{d^4 S}{d\eta^4} \right|_{\substack{\xi=0 \\ S=S_0}} = 2\lambda_{02} \left. \frac{\partial^3 t_{12}}{\partial \eta^3} \right|_0, \quad (5.4.37)$$

which reduces to Equation 4.7.34.

Using the expansion for entropy given in Equation 4.4.47, where ξ is replaced by η , one can assert that entropy changes are of fourth-order in the strength of the shock, as measured here by η , which is the same result that was obtained in the initially unstrained case. Further one imposes that entropy must increase across the shock which yields that the right hand side of Equations 5.4.35 or 5.4.37 must be positive.

In terms of the commonly accepted energy function for generalized Mooney materials given in Equation 4.7.41, this physical requirement that entropy must increase across the shock again gives

$$\left. \frac{\partial^2 f}{\partial I^2} \right|_0 > 0, \quad (5.4.38)$$

which is the same inequality that was obtained in the initially unstrained case.

6. ENTROPY INEQUALITIES: ANISOTROPIC MEDIUM-INITIALLY UNSTRAINED AND AT REST

6.1. General Considerations

An anisotropic medium has the property that the principal axes of stress and strain do not coincide. As was indicated in section 3.5 of this work, it was not possible to find another strain measure so as to reduce both the energy equation and Gibb's equation to identities. So another way is to seek a pseudo-stress which is of a completely spatial character. This formulation was developed in section 3.5 of this work and will now be applied in a few cases to obtain entropy inequalities.

Only a partial study of shock wave propagation in an anisotropic elastic medium is presented here. Plane pure longitudinal shock waves for a particular direction are discussed. The mixed type is not discussed here and further a classification of shock waves as only pure longitudinal type and mixed type, which was achieved for the general problem of curved shock waves in an initially unstrained isotropic medium at rest, was not obtained thus far. These aspects are still under study.

Since the medium is assumed to be initially unstrained and at rest, one has

$$u_{oi,j}=0; \quad v_{oi}=0; \quad e_{oij}=0; \quad t_{oij}=0; \quad \rho_o=\rho_{oo} . \quad (6.1.1)$$

From Equation 4.2.13, the compatibility conditions for discontinuities in the displacement gradients are

$$[u_i] = 0; \quad [u_{i,j}] = \xi_i n_j; \quad \left[\frac{\partial u_i}{\partial t}\right] = -G\xi_i. \quad (6.1.2)$$

where

$$\xi_i = (\xi_1, \xi_2, \xi_3) \quad (6.1.3)$$

is the discontinuity vector in the gradient of the displacement vector.

Then the shock conditions, which are obtained from the balance of mass, linear momentum and energy, the discontinuity in the velocity vector, the consistency condition and the generalized Rankine-Hugoniot relation are given in Equations 4.2.19, 4.2.26, 4.2.28, 4.2.29 and 4.2.31. They are:

$$\rho = \rho_{00}(1-\xi_n); \quad v_i = \frac{-G\xi_i}{1-\xi_n}; \quad t_{ji}n_j = \frac{\rho_{00}G^2\xi_i}{1-\xi_n}, \quad (6.1.4)$$

$$\frac{f_1}{\xi_1} = \frac{f_2}{\xi_2} = \frac{f_3}{\xi_3}; \quad f_i = t_{ji}n_j, \quad (6.1.5)$$

$$[U] = \frac{f_i \xi_i}{2(1-\xi_n)}. \quad (6.1.6)$$

From Equation 4.2.24, the jump in the strain tensor becomes

$$2e_{ij} = \xi_i n_j + \xi_j n_i - \xi^2 n_i n_j; \quad \xi^2 = \xi_k \xi_k. \quad (6.1.7)$$

As explained earlier a new pseudo-stress tensor was defined for the anisotropic case. From Equation 3.5.52, one

has

$$b_{ik} t_{kj} = \frac{\rho}{\rho_{oo}} \frac{\partial U}{\partial e_{ij}}, \quad (6.1.8)$$

where

$$\Delta b_{ij} = (1-2I+4III)\delta_{ij} + 2(1-2I)e_{ij} + 4e_{ik}e_{kj}, \quad (6.1.9)$$

$$\Delta = 1-2I+4II-8III; \quad \underline{a}\underline{b} = \underline{b}\underline{a} = \underline{\delta}, \quad (6.1.10)$$

and

$$I = e_{ij}; \quad II = \frac{1}{2}(e_{ij}e_{jj} - e_{ij}e_{ji}), \quad (6.1.11)$$

$$III = \det(e_{ij}) = |e_{ij}|. \quad (6.1.12)$$

Further it is noted that if one premultiplies Equation 6.1.8 by $a_{mi} = \delta_{mi} - 2e_{mi}$, one obtains

$$t_{mi} = \frac{\rho}{\rho_{oo}} (\delta_{mi} - 2e_{mi}) \frac{\partial U}{\partial e_{ij}}, \quad (6.1.13)$$

which is the last of Equation 3.4.13.

Use of Equation 6.1.8 in Gibb's equation, which is given in Equation 3.5.4, yields

$$dU = \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho} b_{ik} t_{kj} de_{ij}. \quad (6.1.14)$$

Now it was also indicated earlier that the dependence of U on the strain invariants for the general case of anisotropy depends on the type of anisotropy. Further the dependence of U on e_{ij} is only through the invariants

appropriate to the type of anisotropy considered. The type of anisotropy considered here is what is sometimes referred to as transverse isotropy. It can be shown [62, p. 25] that for a transversely isotropic medium the strain energy function is given by [57, 69]

$$U = U(I, II, III, IV, V, S) , \quad (6.1.15)$$

where

$$I = e_{ii} , \quad (6.1.16)$$

$$II = \frac{1}{2}\{e_{ii}e_{jj}-e_{ji}e_{ij}\} , \quad (6.1.17)$$

$$III = \det(e_{ij}) = |e_{ij}| , \quad (6.1.18)$$

$$IV = e_{33} \quad (6.1.19)$$

$$V = e_{13}^2 + e_{23}^2 , \quad (6.1.20)$$

6.2. Plane Pure Longitudinal Shock Waves

From the form of Equations 6.1.4-6.1.18, it can be seen that it is admissible to take $\xi_2=\xi_3=0$ or $\xi_1=\xi_3=0$ or $\xi_1=\xi_2=0$. Hence there are three plane pure longitudinal type shock waves. These can be chosen in general to be defined by

$$\xi_i = (\xi_1, 0, 0) ; \quad n_i = (1, 0, 0) \quad (\text{one-one direction}), \quad (6.2.1)$$

$$\xi_i = (0, \xi_2, 0) ; \quad n_i = (0, 1, 0) \quad (\text{two-two direction}), \quad (6.2.2)$$

$$\xi_i = (0, 0, \xi_3) ; \quad n_i = (0, 0, 1) \quad (\text{three-three direction}). \quad (6.2.3)$$

The plane pure longitudinal shock wave (one-one direction) defined by Equation 6.2.1 is discussed first. Combining Equation 6.2.1 with the first and third of Equation 6.1.4 and Equation 6.1.6, one obtains

$$\rho = \rho_{00}(1 - \xi_1); \quad t_{11} = \frac{\rho_{00} G^2 \xi_1}{1 - \xi_1}, \quad (6.2.4)$$

$$[U] = \frac{t_{11} \xi_1}{2(1 - \xi_1)}, \quad (6.2.5)$$

and the remaining two Equations of motion from the third of Equation 6.1.4 reduce to identities.

The strain tensor, given in Equation 6.1.7, becomes

$$e_{ij} = \begin{pmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.2.6)$$

with

$$e_{11} = \xi_1 - \frac{1}{2} \xi_1^2. \quad (6.2.7)$$

The strain invariants, given in Equation 6.1.16, 6.1.17 and 6.1.18, become

$$I = \xi_1 - \frac{1}{2} \xi_1^2; \quad II=0; \quad III=0, \quad (6.2.8)$$

$$IV = 0; \quad V = 0. \quad (6.2.9)$$

Before obtaining the stress tensor, b_{ij} and Δ , defined

in Equations 6.1.9 and 6.1.10 respectively, will be evaluated for this case. One therefore has

$$\Delta = 1-2I = 1-2e_{11} = (1-\xi_1)^2, \quad (6.2.10)$$

$$b_{11} = \frac{1}{(1-\xi_1)^2}, \quad (6.2.11)$$

$$b_{22} = b_{33} = 1; \quad b_{12} = b_{21} = 0, \quad (6.2.12)$$

$$b_{13} = b_{31} = 0; \quad b_{23} = b_{32} = 0. \quad (6.2.13)$$

The stress tensor and its components can now be obtained from Equation 6.1.13. They are:

$$t_{ij} = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \quad (6.2.14)$$

with

$$t_{11} = \frac{\rho}{\rho_{00}} (1-\xi_1)^2 U_1, \quad (6.2.15)$$

$$t_{22} = \frac{\rho}{\rho_{00}} \{U_1 + e_{11} U_2\}, \quad (6.2.16)$$

$$t_{33} = \frac{\rho}{\rho_{00}} \{U_1 + U_4 + e_{11} U_2\}. \quad (6.2.17)$$

Combining Equations 6.2.6, 6.2.10-6.2.14 and the first of Equation 6.2.4 with Equation 6.1.14, Gibb's equation reduces to

$$\begin{aligned}
 dU &= \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho} b_{11} t_{11} de_{11} \\
 &= \rho_{oo} \theta dS + \frac{t_{11} de_{11}}{(1-\xi_1)^3}, \quad (6.2.18)
 \end{aligned}$$

or

$$\begin{aligned}
 \dot{U} &= \rho_{oo} \theta \dot{S} + \frac{t_{11} \dot{e}_{11}}{(1-\xi_1)^3} \\
 &= \rho_{oo} \theta \dot{S} + \frac{t_{11}}{(1-\xi_1)^2}, \quad (6.2.19)
 \end{aligned}$$

where the dot represents the total derivative with respect to the shock-strength, as measured here by ξ_1 .

From Equation 6.2.5, the generalized Rankine-Hugoniot relation becomes

$$2\dot{U} = \frac{\dot{t}_{11}\xi_1}{(1-\xi_1)} + \frac{t_{11}}{(1-\xi_1)} + \frac{t_{11}\xi_1}{(1-\xi_1)^2} \quad (6.2.20)$$

Equating values of \dot{U} from Equations 6.2.19 and 6.2.20 and simplifying, one obtains

$$2\rho_{oo}\theta\dot{S} = \frac{\dot{t}_{11}\xi_1}{1-\xi_1} - \frac{t_{11}}{(1-\xi_1)^2}. \quad (6.2.21)$$

Since it is assumed stress vanishes with strain, then t_{11} vanishes with ξ_1 . Thus evaluating Equation 6.2.21 at $\xi_1=0$ yields

$$\dot{S}_0 = 0, \quad (\rho_{oo}, \theta_0 \neq 0), \quad (6.2.22)$$

where the suffix "o" means evaluation at $\xi_1=0$ and $S(0)=S_{oo}$.

Differentiating Equation 6.2.21 with respect to ξ_1 yields

$$2\rho_{oo}(\dot{\theta}\dot{S}+\theta\ddot{S}) = \frac{\ddot{t}_{11}\xi_1}{1-\xi_1} - \frac{2t_{11}}{(1-\xi_1)^3} . \quad (6.2.23)$$

Evaluating at $\xi_1=0$ and noting \dot{S} and t_{11} vanishes with ξ_1 yields

$$\ddot{S}_o = 0, \quad (\rho_{oo}, \theta_o \neq 0) . \quad (6.2.24)$$

A final differentiation of Equation 6.2.23 with respect to ξ_1 yields

$$\begin{aligned} 2\rho_{oo}(\ddot{\theta}\dot{S}+2\dot{\theta}\ddot{S}+\theta\ddot{\ddot{S}}) &= \frac{\ddot{\ddot{t}}_{11}\xi_1}{1-\xi_1} + \frac{\ddot{\ddot{t}}_{11}}{1-\xi_1} + \frac{\ddot{\ddot{t}}_{11}\xi_1}{(1-\xi_1)^2} - \frac{2\ddot{\ddot{t}}_{11}}{(1-\xi_1)^3} \\ &\quad - \frac{6t_{11}}{(1-\xi_1)^4} . \end{aligned} \quad (6.2.25)$$

Evaluating at $\xi_1=0$ and noting that \dot{S} , \ddot{S} and t_{11} vanish with ξ_1 yields

$$2\rho_{oo}\theta_o \left. \frac{d^3S}{d\xi_1^3} \right|_{\substack{\xi_1=0 \\ S=S_{oo}}} = \{\ddot{\ddot{t}}_{11}-2\ddot{\ddot{t}}_{11}\}_o . \quad (6.2.26)$$

Now since \dot{S} and \ddot{S} vanish with ξ_1 , the total derivatives in Equation 6.2.26 become partial derivatives with S held constant. Hence Equation 6.2.26 becomes

$$2\rho_{oo}\theta_o \left(\frac{d^3 S}{d\xi_1^3} \right)_{\substack{\xi_1=0 \\ S=S_{oo}}} = \left(\frac{\partial^2 t_{11}}{\partial \xi_1^2} - 2 \frac{\partial t_{11}}{\partial \xi_1} \right)_{\substack{\xi_1=0 \\ S=S_{oo}}} , \quad (6.2.27)$$

which is exactly the same as Equation 4.4.32.

The total derivatives of t_{11} with respect to ξ_1 can be replaced by partial derivatives of the internal energy function U with respect to the strain invariants. Differentiating Equation 6.2.15 twice with respect to ξ_1 and evaluating at $\xi_1=0$ yields

$$\dot{t}_{11}{}_o = \dot{U}_1{}_o , \quad (6.2.28)$$

$$\ddot{t}_{11}{}_o = \{\ddot{U}_1 - 6\dot{U}_1\}{}_o . \quad (6.2.29)$$

Using Equations 6.2.8 and 6.2.9, it can be shown that

$$\dot{U}_1{}_o = \{U_{11}\dot{I}\}{}_o , \quad (6.2.30)$$

$$\ddot{U}_1{}_o = \{U_{111}\dot{I}^2 + U_{11}\ddot{I}\}{}_o . \quad (6.2.31)$$

Combining Equations 6.2.7 and 6.2.8, one further has

$$\dot{I}_o = 1 ; \quad \ddot{I}_o = -1 . \quad (6.2.32)$$

Combining Equations 6.2.28-6.2.32 with Equation 6.2.26, one has the final result

$$2\rho_{00}\theta_0\left.\frac{d^3s}{d\xi_1^3}\right)_{\substack{\xi_1=0 \\ s=s_{00}}} = \{U_{111}-9U_{11}\}_0, \quad (6.2.33)$$

which is the same as Equation 4.4.46.

Use of the expansion for entropy given in Equation 4.4.47, where ξ is replaced by ξ_1 , one can assert that entropy changes are of third-order in the shock-strength, as measured here by ξ_1 , which is the same result that was obtained in the initially unstrained and initially strained isotropic cases. Further one imposes that entropy must increase across the shock wave, which yields, from Equation 6.2.33, that

$$\{U_{111}-9U_{11}\}_0 > 0, \quad (6.2.34)$$

from which an inequality between the material properties in the unstrained natural state can be obtained; as was done in the initially unstrained isotropic case.

It is obvious that the plane pure longitudinal shock wave (two-two direction) defined by Equation 6.2.2 is exactly the same as the one-one direction shock wave. This is true for any pure longitudinal shock wave in the 1-2 plane due to the nature of the anisotropy considered; namely, transversely isotropic. However, the plane pure longitudinal shock wave (three-three direction) defined by Equation 6.2.3 gives a slightly different result. This case is now considered.

Combining Equation 6.2.3 with the first and third of

Equation 6.1.4 and Equation 6.1.6, one obtains

$$\rho = \rho_{00}(1-\xi_3); \quad t_{33} = \frac{\rho_{00}G^2\xi_3}{1-\xi_3} \quad , \quad (6.2.35)$$

$$[U] = \frac{t_{33}\xi_3}{2(1-\xi_3)} \quad . \quad (6.2.36)$$

The strain tensor, given in Equation 6.1.7, becomes

$$e_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_{33} \end{pmatrix} , \quad (6.2.37)$$

with

$$e_{33} = \xi_3 - \frac{1}{2}\xi_3^2 \quad . \quad (6.2.38)$$

The strain invariants given in Equation 6.1.16, 6.1.17 and 6.1.18, take the form

$$I = e_{33} ; \quad IV = e_{33} , \quad (6.2.39)$$

$$II = 0 ; \quad III = 0 ; \quad V = 0 . \quad (6.2.40)$$

The quantities b_{ij} and Δ , which are needed later, given in Equations 6.1.9 and 6.1.10 respectively, reduce to

$$\Delta = (1-\xi_3)^2 ; \quad b_{33} = \frac{1}{(1-\xi_3)^2} , \quad (6.2.41)$$

$$b_{11} = b_{22} = 1 ; \quad b_{12} = b_{21} = 0 \quad , \quad (6.2.42)$$

$$b_{13} = b_{31} = 0 ; \quad b_{23} = b_{33} = 0 \quad . \quad (6.2.43)$$

The stress tensor and its components can now be obtained from Equation 6.1.13. They are:

$$t_{ij} = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix} , \quad (6.2.44)$$

with

$$t_{11} = \frac{\rho}{\rho_{oo}} (U_1 + e_{33} U_2) , \quad (6.2.45)$$

$$t_{22} = \frac{\rho}{\rho_{oo}} (U_1 + e_{33} U_2) , \quad (6.2.46)$$

$$t_{33} = \frac{\rho}{\rho_{oo}} (1 - \xi_3)^2 (U_1 + U_4) . \quad (6.2.47)$$

Combining Equations 6.2.37, 6.2.38, 6.2.41-6.2.44 and the first of Equation 6.2.35, Gibb's equation reduces to

$$\begin{aligned} dU &= \rho_{oo} \theta dS + \frac{\rho_{oo}}{\rho} b_{33} t_{33} de_{33} \\ &= \rho_{oo} \theta dS + \frac{t_{33} de_{33}}{(1 - \xi_3)^3} , \end{aligned} \quad (6.2.48)$$

or

$$\dot{U} = \rho_{oo} \theta \dot{S} + \frac{t_{33} \dot{e}_{33}}{(1 - \xi_3)^3} ,$$

$$= \rho_{00} \theta \dot{\xi} + \frac{t_{33}}{(1-\xi_3)^2} , \quad (6.2.49)$$

where the dot represents the total derivative with respect to the shock-strength, as measured here by ξ_3 .

From Equation 6.2.36, the generalized Rankine-Hugoniot relation becomes

$$2\dot{U} = \frac{\dot{t}_{33}\xi_3}{1-\xi_3} + \frac{t_{33}}{1-\xi_3} + \frac{t_{33}\xi_3}{(1-\xi_3)^2} . \quad (6.2.50)$$

Equating values of \dot{U} from Equations 6.2.49 and 6.2.50 and simplifying, one obtains

$$2\rho_{00}\theta\dot{S} = \frac{\dot{t}_{33}\xi_3}{1-\xi_3} - \frac{t_{33}}{(1-\xi_3)^2} . \quad (6.2.51)$$

Since it is assumed that stress vanishes with strain t_{33} vanishes with ξ_3 . Thus evaluating Equation 6.2.51 at $\xi_3=0$ yields

$$\dot{S}_0 = 0, \quad (\rho_{00}, \theta_0 \neq 0) , \quad (6.2.52)$$

where the suffix "o" means evaluation at $\xi_3=0$ and $S(0)=S_{00}$.

Differentiating Equation 6.2.51 with respect to ξ_3 yields

$$2\rho_{00}(\theta\dot{S}+\theta\ddot{S}) = \frac{\dot{t}_{33}\xi_3}{1-\xi_3} - \frac{2t_{33}}{(1-\xi_3)^3} . \quad (6.2.53)$$

Evaluating at $\xi_3=0$ and noting that \dot{S} and t_{33} vanish with ξ_3 , one obtains

$$\ddot{S}_O = 0, \quad (\rho_{OO}, \theta_O \neq 0) . \quad (6.2.54)$$

A final differentiation of Equation 6.2.53 with respect to ξ_3 yields

$$\begin{aligned} 2\rho_{OO}(\ddot{\theta}\dot{S} + 2\dot{\theta}\ddot{S} + \ddot{\theta}\ddot{S}) &= \frac{\ddot{t}_{33}\xi_3}{1-\xi_3} + \frac{\ddot{t}_{33}}{1-\xi_3} + \frac{\ddot{t}_{33}\xi_3}{(1-\xi_3)^2} \\ &\quad - \frac{2\dot{t}_{33}}{(1-\xi_3)^3} - \frac{6t_{33}}{(1-\xi_3)^4} . \end{aligned} \quad (6.2.55)$$

Evaluating at $\xi_3=0$ and noting \dot{S} , \ddot{S} , and t_{33} vanish with ξ_3 yields

$$2\rho_{OO}\theta_O \left(\frac{d^3 S}{d\xi_3^3} \right)_O = \{\ddot{t}_{33} - 2\dot{t}_{33}\}_O , \quad (6.2.56)$$

Now since \dot{S} and \ddot{S} vanish with ξ_3 , the total derivatives in Equation 6.2.56 become partial derivatives with S held constant. Hence Equation 6.2.56 becomes

$$2\rho_{OO}\theta_O \left(\frac{d^3 S}{d\xi_3^3} \right)_{\substack{\xi_3=0 \\ S=S_{OO}}} = \left(\frac{\partial^2 t_{33}}{\partial \xi_3^2} - 2 \frac{\partial t_{33}}{\partial \xi_3} \right)_{\substack{\xi_3=0 \\ S=S_{OO}}} . \quad (6.2.57)$$

The total derivatives of t_{33} in Equation 6.2.56 can be replaced by partial derivatives of the internal energy function with respect to the strain invariants. Differentiating Equation 6.2.47 twice with respect to ξ_3 and evaluating at $\xi_3=0$ yields

$$\dot{t}_{33})_0 = \{\dot{U}_1 + \dot{U}_4\}_0, \quad (6.2.58)$$

$$\ddot{t}_{33})_0 = \{\ddot{U}_1 + \ddot{U}_4 - 6(\dot{U}_1 + \dot{U}_4)\}_0. \quad (6.2.59)$$

Further, since $U_1 = U_1(I, IV, S)$ and $U_4 = U_4(I, IV, S)$ and noting that \ddot{S} vanishes with ξ_3 , it can be shown that

$$\dot{U}_1)_0 = \{U_{11}\dot{I} + U_{14}\dot{IV}\}_0, \quad (6.2.60)$$

$$\ddot{U}_1)_0 = \{U_{111}\dot{I}^2 + U_{11}\ddot{I} + 2U_{114}\dot{I}\dot{IV} + U_{144}\dot{IV}^2 + U_{14}\ddot{IV}\}_0, \quad (6.2.61)$$

$$\dot{U}_4)_0 = \{U_{14}\dot{I} + U_{44}\dot{IV}\}_0, \quad (6.2.62)$$

$$\ddot{U}_4)_0 = \{U_{444}\dot{IV}^2 + U_{44}\ddot{IV} + 2U_{144}\dot{I}\dot{IV} + U_{114}\dot{I}^2 + U_{14}\ddot{I}\}_0, \quad (6.2.63)$$

From Equations 6.2.38 and 6.2.39, it can be shown that

$$\dot{I}_0 = \dot{IV}_0 = 1; \quad \ddot{I}_0 = \ddot{IV}_0 = -1. \quad (6.2.64)$$

Combining Equations 6.2.58-6.2.64 with Equation 6.2.56, one finally has

$$2\rho_{00}\theta_0 \left. \frac{d^3 S}{d\xi_3^3} \right|_{\substack{\xi_3=0 \\ S=S_{00}}} = \{U_{111} - 9(U_{11} + 2U_{14} + U_{44}) + 3(U_{114} + U_{144}) + U_{444}\}_0, \quad (6.2.65)$$

which reduces to result obtained for the case of an initially unstrained isotropic medium at rest, given in Equation 4.4.46.

Then using the expansion for entropy given in Equation 4.4.47, where ξ is replaced by ξ_3 , one can assert that entropy changes are of third-order in the shock-strength, as measured here by ξ_3 . Further one imposes that entropy must increase across the shock wave, which yields from Equation 6.2.65, that

$$\{U_{111} - 9(U_{11} + 2U_{14} + U_{44}) + 3(U_{114} + U_{144}) + U_{444}\}_0 > 0, \quad (6.2.66)$$

from which an inequality between the material properties of the medium, which is transversely isotropic, can be obtained.

7. DISCUSSION OF THE RESULTS, SUMMARY AND PROSPECTS

The main goal of this study has been threefold: 1. To stress the need of the study of shock waves in a spatial system. 2. To obtain a completely spatial formulation for the analysis of shock waves in a hyperelastic medium; this necessarily led to the formulation of a completely spatial constitutive law which reduces both the energy equation and Gibb's equation to identities. 3. Through such an exact nonlinear elastic formulation of shock waves, attempt to justify the commonly accepted "gas-dynamic" model of shock waves in solids. Further, applications to a number of cases of entropy change for small but finite amplitude shock waves are given.

Formulation of a completely spatial constitutive law which reduces both the energy equation and Gibb's equation to identities in the isotropic case, led to the introduction of principal stresses and strains as fundamental; this need came about from purely thermodynamic considerations. Although this increased the mathematical manipulations, it has made the formulation conceptually clearer and unique. It also has an element of simplicity since there are only three stresses and strains instead of six. Further the formulation has put the Hencky strain measure in a curiously

attractive position based on purely thermodynamic considerations. It is coincidental that in the current literature [65], one finds this strain measure as a strong contender in the search for a strain measure which gives a linear stress-strain law for very large strains.

For an isotropic hyperelastic medium, initially unstrained and at rest, it was shown that there exists only two types of shock waves: the pure longitudinal one and the mixed transverse-longitudinal one, both being dilatational. This was shown to be true for three-dimensional curved shocks as well. For the pure longitudinal shock wave, the entropy is proportional to the curvature of the tension-stretch curve and to the third-order in the strength of the shock. This has complete analogy with the gas-dynamical case, at least for the initially unstrained case. Since the internal energy function and the stress (t_{11}) are both functions of density and entropy only, more general proofs, valid for arbitrary strengths, available in gas-dynamics can be argued in this case [1, 30]. The mixed transverse-longitudinal shock wave (which is also mixed shear-dilatational) which is characterized by a strong transverse mode and a weaker longitudinal mode, has brought out some novel features. Entropy changes for the mixed type were shown to be of fourth-order in the shock-strength. This mixed type forces the second strain invariant, II , to be of

the same order as the first invariant I ; which in turn forces the so called fourth-order elasticity (p) to be equally as important as the second-order elasticity (m). The importance of the elasticity p in the nonlinear shear case does not appear to have brought out in the literature until now. Thus the conventional way of ordering the elasticities in the strain energy function is meaningless, at least in this mixed transverse-longitudinal case.

It was shown that for the initially unstrained case, the pure longitudinal shock wave and the gas-dynamical one are formally the same, at least at the wave-front, which justifies the use of the gas-dynamical model for elastic shocks by a number of workers [8-12]. Although the pressure is not isotropic, this does not enter the analysis at the shock-front. However, more study is needed in the case of the mixed shock wave.

Shock waves in an incompressible hyperelastic isotropic medium were also considered. For the initially unstrained case, it was shown that there exists only one type of shock wave, the pure transverse shock wave. Entropy changes for this shock were shown to be of fourth-order in the strength of the shock wave. These results were shown to be true for three-dimensional curved shock waves as well as plane shock waves. It was further shown that for some conventionally assumed energy functions, this case leads to an isentropic

shock, which raises questions as to the validity of such an energy function for study of wave propagation.

For the initially strained isotropic case, principal shock waves were discussed. Classification of shock waves as was done in the initially unstrained case was not obtained here. More study is needed for the arbitrarily initially strained case in order to classify shock waves as to which are admissible and which are not admissible in the general case. For the plane principal shocks considered, it was shown, as in the initially unstrained case, that there exists two types of shock waves: the pure longitudinal type and the mixed type, both being dilatational. Although there are two types of shock waves, there are nine possible shock waves of these types. Entropy change across the pure longitudinal shock wave was shown to be proportional to the curvature of the stress-stretch curve and the third-order in the shock-strength, which is the same result that was obtained in the initially unstrained case. For the mixed type, entropy change was shown to be of fourth-order in the strength of the shock, which is again what was obtained in the initially unstrained case.

Plane principle shock waves in an incompressible isotropic medium were also discussed. For this case it was shown there exists one type of principal shock, namely, the pure transverse shock wave, which is again the same result that

was obtained in the initially strained case.

The complete spatial formulation for study of shock waves in an anisotropic hyperelastic medium was also given. Here a pseudo-stress tensor was introduced in order to reduce the energy equation and Gibb's equation to identities. Entropy changes for plane pure longitudinal shock waves in an initially unstrained transversely isotropic medium were obtained. For this type, only three particular types were studied. It was shown that the entropy change was of third-order in the strength of the shock for all three cases.

As this study indicates, a more thorough investigation is needed in the cases of an arbitrarily strained isotropic medium and an initially unstrained or initially strained anisotropic medium, in order to classify the types of shocks that are admissible and are not admissible.

Due to the present state of lack of knowledge of the form of the internal energy function, it was necessary to resort to approximations; e.g., polynomial expansions. So the results obtained here are limited to small but finite amplitude (strength) shock waves.

In conclusion, some aspects still need further study but is hoped that the present study has contributed to the understanding of shock waves by posing the problem in full generality and providing some interesting solutions.

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10. APPENDIX

10.1. Evaluation of the Third-Order Elasticities ℓ , m , and n

It is first intended here to transform the conventional third-order elasticities ℓ , m and n due to Murnaghan [70] into the present notation and then to evaluate them from some known experimental data.

Now, the third elasticities ℓ , m and n were originally defined by Murnaghan [70, p. 63] via a strain energy function expansion as

$$U = \frac{1}{2}(\lambda_E + 2\mu_E)I_E^2 - 2\mu_E II_E + \frac{1}{3}(\ell_E + 2m_E)I_E^3 - 2m_E I_E II_E + n_E III_E + \dots, \quad (10.1.1)$$

where¹¹ I_E , II_E and III_E are the invariants of the Lagrangian strain tensor, defined as (Equation 3.1.4)

$$2E_{AB} = C_{AB} - \delta_{AB}. \quad (10.1.2)$$

The strain energy function expansion in the present notation is given in Equation 4.4.53 as (with only corresponding terms included)

$$U = \frac{1}{2}(\lambda + 2\mu)I^2 - 2\mu II + \ell I^3 + m I II + n III + \dots, \quad (10.1.3)$$

where I , II and III are the invariants of the Eulerian strain tensor defined as

¹¹The subscript "E" is not an index. It is used here to designate quantities defined with respect to the Lagrangian strain tensor E_{AB} .

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i} u_{k,j} \quad . \quad (10.1.4)$$

From [59, p. 32], it is easily shown that

$$DI_E = I - 4II + 12III \quad , \quad (10.1.5)$$

$$DII_E = II - 6III \quad , \quad (10.1.6)$$

$$DIII_E = III \quad , \quad (10.1.7)$$

where

$$D = 1 - 2I + 4II - 8III \quad . \quad (10.1.8)$$

Substituting Equations 10.1.5-10.1.8 into Equation 10.1.1 yields

$$\begin{aligned} U = & \frac{1}{2}(\lambda_E + 2\mu_E) \frac{(I - 4II + 12III)^2}{D^3} - 2\mu_E \frac{(II - 6III)}{D} \\ & + \frac{1}{3}(\lambda_E + 2\mu_E) \frac{(I - 4II + 12III)^2}{D^3} - 2\mu_E \frac{(I - 4II + 12III)(II - 6III)}{D^2} \\ & + n_E \frac{III}{D} + \dots \quad . \end{aligned} \quad (10.1.9)$$

If, after division by D , one retains terms $O(II)$ only, Equation 10.1.9 reduces to

$$\begin{aligned} U = & \frac{1}{2}(\lambda_E + 2\mu_E) (I^2 - 8I II) - 2\mu_E (II + 2I II) \\ & + \frac{1}{3}(\lambda_E + 2\mu_E) I^3 - 2\mu_E II + n_E III + \dots \quad . \end{aligned} \quad (10.1.10)$$

Equating coefficients of like powers of the strain invariants in Equations 10.1.3 and 10.1.10 gives the result

$$\lambda = \lambda_E ; \quad \mu = \mu_E ; \quad n = n_E , \quad (10.1.11)$$

$$\ell = \frac{1}{3}(\ell_E + 2m_E) , \quad (10.1.12)$$

$$m = -2(m_E + 2\lambda_E + 6\mu_E) . \quad (10.1.13)$$

Seeger in [67, p. 247] gives the following experimental data for iron:

$$\mu_E = 8.26 \cdot (10)^3 \text{ kg/mm}^2 ; \quad k_E = 17.0 \cdot (10)^3 \text{ kg/mm}^2 , \quad (10.1.14)$$

$$\frac{\ell_E}{\mu_E} = -1.6 ; \quad \frac{m_E}{\mu_E} = -10.1 ; \quad \frac{n_E}{\mu_E} = -22.7 , \quad (10.1.15)$$

where k_E is the bulk modulus and

$$\lambda_E = \frac{1}{3}(3k_E - 2\mu_E) . \quad (10.1.16)$$

Substituting Equation 10.1.14 into Equation 10.1.16 yields

$$\frac{\lambda_E}{\mu_E} = 1.39 . \quad (10.1.17)$$

Use of Equations 10.1.15 and 10.1.17 in Equations 10.1.11, 10.1.12 and 10.1.13 gives the final result

$$\frac{\ell}{\mu} = -7.28 ; \quad \frac{m}{\mu} = 2.64 , \quad (10.1.18)$$

$$\frac{n}{\mu} = -22.7 , \quad (10.1.19)$$

which are the third-order elasticities in the present notation for the experimental data given in Equations 10.1.14

and 10.1.15.

It is worth while to note, however, that Seeger, as quoted by Truesdell [60, p. 230], remarks that different interpretations of the same experimental data for these third-order elasticities lead to somewhat different values. It is further pointed out there, that the literature of the subject multiplies.

Therefore, it appears that numerical values of these are to be taken with some reservations and are given as typical illustrations for qualitative comparison only.

10.2. Derivation of the Rivlin-Saunders Strain Energy Function

Experiments of Rivlin-Saunders [59, p. 213] have suggested the following strain energy function for an incompressible isotropic hyperelastic material (which is argued as valid for a wider range of deformation than previous ones),

$$U_{c^{-1}} = \alpha(I_{c^{-1}}^{-3}) + f(II_{c^{-1}}^{-3}) , \quad (10.2.1)$$

where¹² $I_{c^{-1}}$ and $II_{c^{-1}}$ are the invariants of the left Cauchy-Green strain tensor defined as (Equation 3.1.5)

$$(c^{-1})_{ij} = x_{i,A} x_{j,A} , \quad (10.2.2)$$

¹²The subscript " c^{-1} " is not an index. It is used here to designate quantities defined with respect to the left Cauchy-Green strain tensor $(c^{-1})_{ij}$.

and f is a function of II alone.

It is desired here to transform Equation 10.2.1 into the present notation.

Now, incompressibility imposes the following constraint (Equation 3.5.40)

$$\frac{\rho}{\rho_{00}} = 1 = \{1-2I+4II-3III\}^{1/2} , \quad (10.2.3)$$

or

$$1 = 1-2I+4II-8III . \quad (10.2.4)$$

From [59, pp. 31-32], the relationships between the invariants of the left Cauchy-Green tensor and the Eulerian strain tensor are found to be

$$DI_{C-1} = 3-4I+4II , \quad (10.2.5)$$

$$DII_{C-1} = 3-2I , \quad (10.2.6)$$

where

$$D = 1-2I+4II-8III . \quad (10.2.7)$$

But by Equation 10.2.4, $D=1$. Therefore Equations 10.2.5 and 10.2.6 reduce to

$$I_{C-1} = 3-4I+4II , \quad (10.2.8)$$

$$II_{C-1} = 3-2I . \quad (10.2.9)$$

Substituting Equations 10.2.8 and 10.2.9 into Equation 10.2.1 yields the final result

$$U = \alpha(-4I+4II)+f(-2I) \quad ,$$

or

$$U = 4\alpha(II-I)+f(-2I) \quad . \quad (10.2.10)$$

An interesting feature of Equation 10.2.10 is that the arbitrary function, f , is a function of the first invariant, I , only. As previously mentioned, based perhaps on experimental evidence, this arbitrary function is taken to be a slowly varying function of its argument. Further, it is often taken as a constant (small) times its argument. Written in terms of the present notation, as shown above, this argument only involves the first invariant. However, for linear theory, the constraint imposed by incompressibility, given in Equation 10.2.4, yields

$$I = I - 2I \quad , \quad (10.2.11)$$

where terms of order higher than I are neglected. Thus one gets

$$I = 0. \quad (10.2.12)$$

It is also noted that quadratic terms in the definition of strain are also neglected in the linear theory.